INTERMEDIATE HYDROSTATICS

BY

E. E. PREIDEL, M.Sc., A.R.C.S., D.I.C.

Lecturer in Mathematics, Imperial College, London

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PREFACE

THIS book has been written for science and engineering students in schools and universities with two objects in view: to show how the fundamental principles of Hydrostatics are applied in the various branches of Engineering, and to cover the requirements in Hydrostatics of the various Higher School Certificate examinations, and of University Scholarship and Intermediate Degree examinations.

No apology is now needed for using elementary calculus in various proofs, although alternative methods are generally also given. A large number of worked examples have been included to give adequate illustration of the principles involved.

Acknowledgement is due to the Senate of the University of London for permission to use questions set at their Higher School Certificate and Intermediate Arts, Science, and Engineering examinations.

E. E. P.

June 1948.

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INTERMEDIATE HYDROSTATICS

CHAPTER I

FLUID PRESSURE

1. Hydrostatics

Hydrostatics is the name given to the study of the equilibrium of fluids at rest and to the forces maintaining equilibrium. As the name implies, it is a branch of Statics dealing originally with water, a typical liquid, and later extended to cover all fluids. The subject is believed to have originated with the work of Archimedes (250 B.C.), but it was not until 1653 that the fundamental principles of the theory of hydrostatics were clearly explained and illustrated by the French philosopher and mathematician, Pascal. These principles have a very wide field of practical application; first, because many practical appliances such as pumps, hydraulic presses and turbines function by reason of them, and secondly, because they form the basis of the subject of *hydrodynamics* which deals with the *motion* of fluids under the action of forces and the motion of solids through fluids, and consequently embraces such practical problems as the flow of liquids through pipes, the shape of the section of aeroplane wings, the stability and shape of ships, and many others. The parts of the subject which are of particular importance to the practical engineer are termed *Hydraulics*. Some of the appliances dependent on hydrostatical principles are considered in Chapter XI.

We begin our study of Hydrostatics by examining more closely what we mean by such words as fluid, liquid and solid.

2. The three forms of matter

All matter may be classified into two groups, *solid and fluid* and, as the name implies, a fluid is a substance which *flows* or is capable of *flowing*. Fluids (of which water and air are the two most common) may be subdivided into *liquids* (*e.g.* water) and *gases* (*e.g.* air), and it is important to notice some of the characteristics of each of the three groups.

A *solid* tends to keep its same shape for an indefinite length of time and consequently offers considerable resistance to any force tending to change its shape or volume.

A *liquid* is a fluid which offers very little resistance to a force tending to change its shape, but considerable resistance to a force tending to

change its volume, i e it does not change its volume very much for variations in pressure, but it readily assumes the shape of the vessel in which it is contained

A *gas* is a fluid which is readily compressed and changes in volume with change of pressure If a quantity of gas is introduced into a closed vessel it will expand and fill the vessel and consequently will have no free surface

These distinguishing characteristics are conveniently summed up in the words —

A *solid* has both size and shape,
A *liquid* has size but not shape,
A *gas* has neither size nor shape

All the chemical elements and many other substances can exist

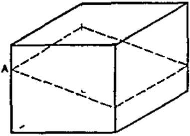


Fig 1

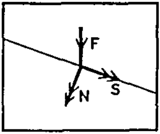


Fig 2

in all three forms, depending upon external conditions Everyone is familiar with ice, water and steam, different forms of the same substance produced by variations in temperature It was not, however, until comparatively recently (1878) that air, oxygen and nitrogen were liquefied by the French physicists Cailletet and Pictet Although the three forms are distinct, the change from one to another is gradual, during which the substance may exhibit some of the characteristics of both the neighbouring states

3 Solids and fluids

Since we are concerned in this book with fluids we must consider a little more deeply the essential differences between solids and fluids

Suppose we have a solid block of wood divided by an imaginary plane AB into an upper part and a lower part (Fig 1) The upper part exerts downward forces on the lower part by reason of its weight and by the methods of Statics (see *Tutorial Statics*, p 24) these downward forces may be resolved into forces at right angles to the plane section and those parallel to this section as shown in the sectional

diagram (Fig. 2). The forces N constitute a normal thrust across the section, and the forces S are termed *shearing forces.* The directions as shown are those of the component forces exerted by the upper part of the block on to the lower and in consequence the lower part exerts equal and opposite forces on the upper part by Newton's Third Law of equal action and reaction. All these combined forces are called the *stress* at the section, while the shearing forces and their counterbalancing forces alone constitute the *shearing stress* at the section. It is clear that the solid block can withstand the shearing stress indicated above and it does this by virtue of its *rigidity.* Since it also withstands the normal thrust across any section we arrive at the following definition. A *solid* is a body which offers permanent resistance to any form of stress, providing the stress is not too great.

It is obvious that if too great a stress be applied the solid may break, but it is the permanent resistance which it offers which distinguishes it from a fluid, for *a fluid is incapable of withstanding permanently any shearing stress,* however small. It is obvious from experience that if, in Fig. 1, the upper part is imagined to be fluid, no equilibrium is possible, the fluid deforms and runs down—it has no rigidity.

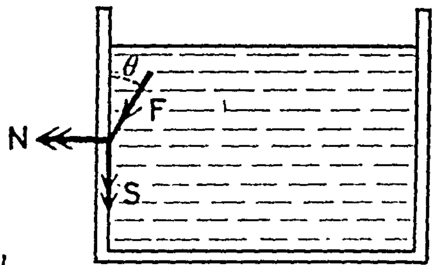


Fig. 3.

In the above definition of a fluid, the reason for the qualification "permanently" is that some fluids are sluggish or *viscous* and may require time before motion is produced by a shearing stress, but however viscous a fluid may be, a shearing stress, although small, will *eventually* produce motion so that the equilibrium of a fluid under shearing stress is impossible. On the other hand, a fluid of low viscosity flows rapidly under such a stress. Since we shall only be dealing with fluids at rest we shall not be concerned with viscosity, we need only to realise that in any fluid in equilibrium there can be *no shearing stress.*

4. Fundamental property of a fluid

From the fact that a fluid possesses no rigidity we may deduce the following fundamental property of a fluid:—*when a fluid is in equilibrium, the force which it exerts on any surface with which it is on contact, is at right angles to that surface.*

Let Fig. 3 represent a section of a vessel containing a fluid, and suppose that F represents the force which the fluid exerts on a small

area in the side of the vessel We are not concerned here with the magnitude of the force but only its direction Let the direction of F make an angle θ with the vertical Then F may be resolved into its two components N and S at right angles so that

$$S = F \cos \theta$$

and $\quad N = F \sin \theta$ (see *Tutorial Statics*, p 24)

But the force S is a shearing force which would make the particles of fluid slide down the walls of the vessel Thus motion would ensue which is contrary to our original statement that the fluid is in equilibrium Hence for equilibrium S must equal zero so that

$$F \cos \theta = 0$$

Thus either $F = 0$ or $\cos \theta = 0$

We know by experience that F does not equal zero because if a portion of the side of the vessel be removed the fluid will run out so that there is certainly some force F Thus $\cos \theta = 0$ i e $\theta = 90°$ and the force F is therefore wholly perpendicular to the surface

5 Perfect fluid

It is often convenient to imagine a fluid such that *whether at rest or in motion* the force exerted by it on any surface with which it is in contact is at right angles to that surface Such a hypothetical fluid is called a *perfect fluid* and in consequence of this definition there can be no force in the nature of friction between the fluid and a surface in contact with it This idea of a perfect fluid becomes of much more importance in the subject of Hydrodynamics it does not greatly concern us in Hydrostatics since as we have seen if a fluid is in equilibrium whether it be very viscous or not it always exerts a force at right angles to any surface with which it is in contact It is, however, worth while to notice the value of such a conception At first sight it might appear illogical to take as a perfect fluid an imaginary fluid which does not exist in Nature especially as an aid to the formulation of practical laws but the underlying theoretical laws governing the flow of such a fluid may be fairly easily established and then modified to take into consideration the deviations of actual fluids from the ideal of a perfect fluid

6 Pressure of a fluid

Let us consider some experimental evidence on the thrust or push exerted by a fluid

Suppose Fig. 4 represents a section of a fluid contained in a rectangular vessel. We know by experience that the fluid will spurt out of a hole at A with more force than it would out of a hole at B, *providing the areas of the holes are the same,* showing that the fluid exerts a greater thrust on the area at A than at B. Since the area is an important factor in these considerations we denote the *thrust per unit area* by the term *pressure.* If the thrust has the same value over all the parts of our small area, we say that the pressure is uniform. In this case the pressure is not uniform and, as we shall see later (§ 16), it increases uniformly with the depth. Suppose we have a small area, a square feet, in the sides of the vessel, and that the total thrust on this area be N lb. wt. Then the *average pressure* over this area is $\frac{N}{a}$ lb. wt. per sq. ft. If we imagine the area a to be diminished, the total thrust N is consequently diminished and the limiting value of $\frac{N}{a}$ as a reduces to zero is called the *pressure intensity* at the point to which the area is reduced. It will still be in units of lb. wt. per sq. ft. It is obvious that there is a pressure intensity all round the boundary between the fluid and the vessel, but it is not quite so obvious that there is also a pressure intensity at every point within the fluid. This may be shown as follows. Imagine what would happen if a volume S of the fluid were removed from the interior (Fig. 4): the surrounding fluid would rush in and fill the cavity and thus the fluid which is in fact enclosed by S must exert a counteracting thrust on the surrounding fluid. Thus there are pressure intensities at all points of the boundary of the volume S, and since this may be anywhere there is a pressure intensity at all points of the fluid.

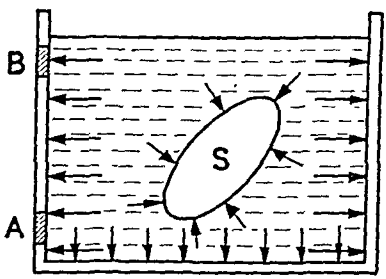


Fig. 4.

We know the direction of the pressure at points where the fluid is in contact with the vessel, but we now require to investigate the nature of the pressure intensity at points within the fluid.

7. The pressure intensity at any point in a fluid at rest is the same in all directions

Consider an imaginary prism drawn in the fluid of triangular section, sides a, b, c inches, the length of the prism being d inches (Fig. 5). The fluid contained by the prism is in equilibrium as a result of the pressure of the liquid outside the prism on the three rectangular faces

and two triangular ends of the prism together with the weight of the fluid in the prism We may consider the pressure on each face to be equivalent to a single force Suppose that the base of the prism is in a horizontal plane, then resolving the forces horizontally along the axis of the prism the forces on the two ends are equal and opposite If we resolve the remaining forces in a plane at right angles to the axis of the prism we obtain valuable information concerning the average pressures over the rectangular faces Let p_1, p_2, p_3 represent the average pressures over these faces in lb wt per square inch and let the fluid weigh w lb wt per cubic inch Then the total force exerted by the fluid outside the prism over the face of sides a d is $p_1 ad$ lb wt, since the area of this face is ad sq in Similarly, the total forces over the other faces are $p_2 bd$, and $p_3 cd$ and the weight of the fluid in the prism (W) is $\frac{1}{2} c\ a \sin\alpha\ d\ w$ These forces are shown in the sectional diagram (Fig 6) Resolving these forces horizontally we have

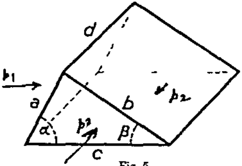


Fig 5

$$p_1 ad \sin\alpha = p_2 bd \sin\beta \qquad \text{(i)}$$

and resolving vertically

$$p_1 ad \cos\alpha + p_2 bd \cos\beta + W = p_3 cd \qquad \text{(ii)}$$

From the geometry of the triangle (Fig 6) the perpendicular height $= a \sin\alpha = b \sin\beta$ and the side $c = a\cos\alpha + b\cos\beta$ Using the first relation in equation (i) we have

$$p_1 = p_2$$

and then equation (ii) becomes

$$p_1 d(a\cos\alpha + b\cos\beta) + W = p_3 cd$$

$$p_1 dc + \tfrac{1}{2} acdw \sin\alpha = p_3 cd$$

$$p_1 + \tfrac{1}{2} aw \sin\alpha = p_3$$

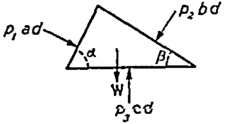


Fig 6

We are interested in the pressure at a *point* in the fluid so we consider what happens in the limiting case when the prism becomes smaller and smaller If the prism shrinks to a line keeping α and β fixed, then as a tends to zero so also must $\frac{1}{2} aw \sin\alpha$ and consequently our last relation becomes in the limiting case $p_1 = p_3$

Also when the triangle reduces to a point the pressures p_1, p_2, p_3 which were average pressures become in the limiting case the pressure intensities at the point, so that our final result is

$$p_1 = p_2 = p_3,$$

and since α and β may have any values we deduce that the pressure intensity in a fluid at rest is the same in all directions.

8. Units

We have already defined pressure as force or thrust per unit area, and in the cases we have considered we took 1 lb. wt. as the unit of force. We might equally well have expressed our pressure in poundals per square inch or per square foot or in dynes per square centimetre. Let us illustrate with one or two examples.

Example 1.—*If a ton of water is contained in a rectangular tank whose base is 4 ft. × 2 ft., the whole thrust on the base*

$$= 1 \text{ ton wt.} = 2240 \text{ lb. wt.}$$

and the area over which it is distributed $= 8$ sq. ft.;

$\therefore$ pressure intensity on base of tank $= \frac{2240}{8} = 280$ lb. wt. per sq. ft.

$= \frac{280}{144} = 1{\cdot}94$ lb. wt. per sq. in.

Example 2.—*If the pressure of the steam inside a boiler is* 140 *lb. wt. per square inch, to find the thrust supported by the ends of the boiler, given that they are circular and* 6 *ft. in diameter.*

Here the area of either end $= \frac{22}{7} \times (\frac{6}{2})^2$ sq. ft. $= \frac{198}{7}$ sq. ft.

$= \frac{28512}{7}$ sq. in.,

and the pressure intensity $= 140$ lb. wt. per sq. in.;

$\therefore$ thrust on either end $= \frac{28512}{7} \times 140 = 28512 \times 20$

$= 570240$ lb. wt. $= 254\frac{4}{7}$ ton wt.

Example 3.—*To express a pressure intensity of* 15 *lb. wt. per square inch in* (i) *ton wt. per square foot,* (ii) *poundals per square foot.*

(i) On 1 sq. in. the pressure produces a thrust of 15 lb. wt.

Therefore, on 1 sq. ft. (= 144 sq. in.) the pressure produces a thrust of 15×144 lb. wt.;

$\therefore$ pressure intensity $= 15 \times 144$ lb. wt. per sq. ft.

$$= \frac{15 \times 144}{2240} \text{ ton wt. per sq. ft.}$$

$= \frac{27}{28}$ of a ton per square foot.

(ii) Taking the acceleration of gravity as 32 ft per sec per sec,

1 lb weight = 32 poundals;

$\therefore$ pressure intensity = $15 \times 144 \times 32$ poundals per square foot

= 69120 poundals per square foot

Example 4.—*To express a pressure intensity of 1000 oz wt per square foot in pounds weight per square inch*

On 1 sq ft (= 144 sq in) the pressure exerts a force = 1000 oz wt

Therefore, on 1 sq in the pressure exerts a force

$$= \frac{1000}{144} \text{ oz wt} = \frac{1000}{144 \times 16} \text{ lb wt};$$

$$\therefore \text{ pressure intensity} = \frac{1000}{144 \times 16} = 0.434028 \text{ lb wt per sq in}$$

Example 5.—*To express a pressure of 1 kilog wt per square metre (i) in grammes weight per square centimetre, (ii) in C G S dynamical units*

On 1 sq metre (= 100^2 sq cm) the pressure exerts a force

1 kilog = 1000 grm wt

Therefore, on 1 sq cm the pressure exerts a force

$$= \frac{1000}{100^2} \text{ grm wt} = 0.1 \text{ grm wt}$$

Hence pressure intensity = 0.1 grm wt per sq cm

The C G S dynamical unit of pressure is a pressure of 1 dyne per square centimetre

Now the acceleration of gravity = 981 cm per sec per sec,

weight of a gramme = 981 dynes,

given pressure intensity = 981×0.1 = 98.1 dynes per sq cm

9 Transmissibility of fluid pressure

When any pressure is applied to any part of the surface of a fluid, an equal and uniform pressure is transmitted over the whole fluid

This principle is also known as Pascal's Law, and we may verify it experimentally as follows Suppose fluid is contained in a closed vessel to which there are attached a number of tubes housing pistons to which forces may be applied (Fig 7), and suppose that for a position of equilibrium the forces required are P, R, S, T lb wt Suppose also that the cross sectional areas of the tubes are a, b, c, d sq in as indicated on the diagram

Suppose that an additional pressure (p lb. wt. per sq. in.) is applied to piston A, *i.e.* an additional force of pa lb. wt.; then it will be found necessary to apply additional forces of pb, pc, pd lb. wt. to R, S and T respectively in order to maintain equilibrium, showing that the pressure against them has been increased by p lb. wt. sq. in. In other words, the additional pressure p applied to A has transmitted itself through the fluid to the other pistons.

This experiment is difficult to set up and we may prove the principle by means of the Conservation of Energy. Suppose the piston A is pushed in a distance x in. by the force of F lb. wt., resulting in piston D being moved out a distance y in. against the force T lb. wt., pistons B and C being fixed in position meanwhile, so that the total volume

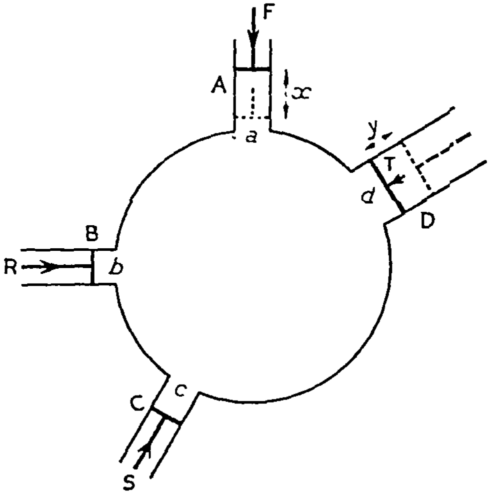


Fig. 7.

remains unaltered. Since a fluid offers no resistance to changes of shape which do not alter its volume, no work is done on the fluid itself by moving the pistons A and D. Therefore the work done by F is equal to the work done against T,

i.e. $$\mathrm{F}x = \mathrm{T}y \quad \text{(i)}$$

Also, since no change in volume has taken place

$$ax = dy \quad \text{(ii)}$$

Dividing (i) by (ii)

$$\frac{\mathrm{F}}{a} = \frac{\mathrm{T}}{d}.$$

But $\frac{\mathrm{F}}{a}$ is the pressure (in lb. wt. per sq. in.) on the piston A and $\frac{\mathrm{T}}{d}$ is

the pressure on the piston D Thus the pressures on these pistons are equal Similarly we may prove that the pressures on the other pistons are also equal

10. The Bramah (or Hydraulic) Press

This is a practical application of the principle of transmissibility of fluid pressure considered in the last section

Essentially the Hydraulic Press consists of a wide vertical cylinder connected by a tube near its base to a narrow vertical cylinder (Fig 8) Each cylinder has a piston fitted and the space below the pistons is filled with water The narrower piston (the pump plunger) is driven downwards, giving rise to a movement upwards of the larger piston (the press plunger or ram) which may compress material between itself and a fixed horizontal platform fitted above What at first

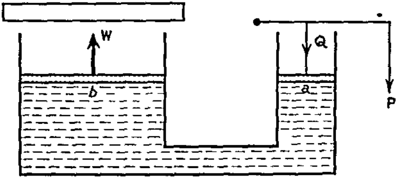


Fig 8

sight may appear surprising is that a *small* force on the pump plunger gives rise to a *larger* force on the press plunger or ram Suppose that the cross sectional areas of the pistons are a b sq in as indicated on the diagram so that b is greater than a Then a downward force of Q lb wt on the pump plunger gives a pressure of $\frac{Q}{a}$ lb wt per sq in which, by 9 above, is transmitted through the liquid to the larger piston If W lb wt is the compressive force which it exerts, then the pressure on the piston is $\frac{W}{b}$ lb wt per sq in, and the pressure is due to, and must be equal to, that of the ram

Thus $$\frac{Q}{a} = \frac{W}{b},$$

so that $$W = \frac{b}{a} Q$$

Since $b > a$, $W > Q$, and the compressive force of W is *greater* than the applied force Q in the ratio of their cross-sectional areas. For instance, if the larger piston has four times the cross-section of the smaller piston, then $b = 4a$, so that $\frac{b}{a} = 4$ and consequently $W = 4Q$, *i.e.* the compressive force is four times as large as the applied force. In practice it is convenient to apply the force Q by means of a lever as shown in the figure. Theoretically, by arranging the ratio of the two pistons we may make a small force support a weight of any magnitude. This was known as the "hydrostatic paradox," but there is nothing paradoxical in it since the same fact applies to the lever and every other simple machine.

A few numerical examples are now included.

Example 1.—*If the pistons of a Bramah press are circular, and of diameters* 1 *in. and* 2 *ft. respectively, to find the force required to overcome a resistance of* 9 *ton wt.*

Here the areas of the circular pistons are respectively $\frac{1}{4}\pi \times 1^2$ and $\frac{1}{4}\pi \times 24^2$ sq. in. Also the thrust of the fluid on the larger piston is required to be 9×2240 lb. wt.

Hence the pressure of the fluid is

$$\frac{9 \times 2240}{144\pi} \text{ lb. wt. per sq. in.,}$$

and since the area of the smaller piston is $\frac{1}{4}\pi$ sq. in., the force on it must be

$$= \frac{9 \times 2240 \times \frac{1}{4}\pi}{144\pi} = \frac{9 \times 560}{144} = 35 \text{ lb. wt.}$$

Example 2.—*If the areas of the two plungers are* $\frac{1}{4}$ *sq. in. and* 10 *sq. in., and the pump-plunger is worked by a lever whose arms are* 2 *in. and* 28 *in., to find the resistance that can be overcome by applying a force of* 15 *lb. to the end of the longer arm of the lever.*

Let Q be the resultant thrust on the plunger. For the equilibrium of the lever, we have, by taking moments about the fulcrum,

$$Q \times 2 = 15 \times 28;$$

$$\therefore \quad Q = 15 \times 14 = 210 \text{ lb. wt.}$$

This force of 210 lb. wt. is distributed over the area of the small plunger, which is $\frac{1}{4}$ sq. in.;

$\therefore$ pressure produced $= 210 \div \frac{1}{4} = 210 \times 4 = 840$ lb. wt. per sq. in.

This pressure is transmitted to the surface of the larger plunger, whose area is 10 sq. in.;

$$\therefore \quad \text{upward thrust on large plunger} = 840 \times 10 \text{ lb. wt.}$$

Hence the press can overcome a resistance of 8400 lb., that is, $3\frac{3}{4}$ ton wt.

Example 3.—*If, in the last example, the end of the lever is raised and lowered through 1 ft at every stroke, to find the number of strokes required to raise the press plunger through 1 in*

Since the arms of the lever are 28 in and 2 in respectively, therefore, when the end of the longer arm is lowered through 1 ft, the pump plunger falls through $\frac{1}{14}$ ft, i.e. $\frac{6}{7}$ in

Hence the volume of water forced out of the pump cylinder

$$\tfrac{1}{4} \times \tfrac{6}{7} \text{ cub in} = \tfrac{3}{14} \text{ cub in}$$

This volume is forced into the press cylinder, hence the press plunger rises at each stroke through $\frac{3}{14} \div 10$ in, i.e. through $\frac{3}{140}$ in

Hence the number of strokes required to raise it through 1 in is

$$= \tfrac{140}{3} = 46\tfrac{2}{3}$$

Hence 46 complete strokes must be made, and the lever must be pressed two thirds down in the 47th stroke

Exercises I

(1) If a gallon of water weighs 10 lb and a cubic foot weighs 1000 oz, how many gallons are there in a cubic foot?

(2) Compare (i.e. find the ratio of) the following pressures —
  (i) 14 lb wt per square inch and 8 ton wt per square yard
  (ii) 28 lb wt per square inch and 16.2 ton wt per square foot
  (iii) 28 grm wt per square centimetre and 16.1 kilog wt per square metre

(3) A piston 6 sq in in area is inserted into one side of a closed cubical vessel measuring 10 ft each way, filled with water, the piston is pressed inwards with a force of 12 lb wt Find the increase of thrust produced on the six sides of the vessel

(4) The neck and bottom of a bottle are $\frac{1}{2}$ in and 4 in in diameter respectively If when the bottle is full of water, the cork is pressed in with a force of 1 lb, what force is exerted upon the bottom of the bottle?

(5) A prism, whose height is 10 mm and whose base is an isosceles triangle with sides 10, 10, 12 mm respectively, is placed in a fluid where the average pressure is 100 grm wt per square centimetre. Find the thrusts on the respective faces, and the ratios to them of the weight of water required to fill the prism

(6) If all the dimensions of the prism (see the last question) be reduced to one-tenth of the above measurements show that these ratios will be one tenth of their previous values Hence show that, if a prism be taken sufficiently small, the weight of the fluid in it can be neglected in comparison with thrusts of the fluid on its faces

(7) In a hydraulic press the pump-plunger is a cylinder 1 cm. in diameter, and makes a stroke 7 cm. long. The plunger of the press is 20 cm. in diameter. Calculate (*a*) the pressure in the press when a weight of 100 lb. is applied to the pump-plunger (ignoring friction); (*b*) the force acting on the press-plunger; (*c*) the number of strokes which the pump must make in order to raise the press-plunger 10 cm.

(8) In the Bramah press the areas of the two cylinders are $\frac{1}{6}$ sq. in. and 5 sq. in., and the lengths of the arms of the lever by which it is worked are 36 in. and $1\frac{1}{8}$ in. How much thrust is obtained by applying to the end of the longer arm a force of 15 lb. ?

(9) In the Bramah press the areas of the two pistons are $\frac{1}{4}$ sq. in. and 16 sq. in. respectively. If the lengths of the arms of the lever are in the ratio of 20 : 1, what force must be applied at the end of the lever in order to produce a thrust of 16,000 lb. wt. ?

(10) The pump plunger of a Bramah press has a diameter of 1 in. and is worked by a lever with a velocity ratio of 10 : 1. The diameter of the ram is 1 ft. and a force of 20 lb. wt. is applied to the handle of the lever. Find the force exerted by the ram, if the efficiency of the press is 70%.

(11) The cross-sectional area of the ram of a hydraulic press is n times that of its piston. If the resistances due to friction are F and R lb. wt. at the piston and ram respectively, show that to raise a weight of W lb. on the ram it is necessary to apply a force to the piston of amount $(W + R + nF)/n$ lb. wt.

(12) Express a pressure of 30 lb. wt. per square inch in kilogrammes per square centimetre, given that 1 lb. = 453·6 grm., and 1 inch = 2·54 cm.

ANSWERS

1. $6\frac{1}{4}$. 2. (i) 81 : 80; (ii) 1 : 9; (iii) 400 : 23.

3. 172,800 lb. wt. 4. 64 lb. wt.

5. 100 grm. wt., 100 grm. wt., 120 grm. wt., 48 grm. wt., 0·0048, 0 0048, 0·004, 0·01.

7. (*a*) $127\frac{3}{11}$ lb. wt. per sq. cm.; (*b*) $17\frac{5}{7}$ ton wt.; (*c*) $571\frac{3}{7}$.

8. 14,400 lb. wt. 9. $12\frac{1}{2}$ lb. wt. 10. 9 ton wt.

12. 2·11 kilog. per sq. cm.

CHAPTER II

DENSITY AND SPECIFIC GRAVITY

11. Mass and weight

The distinction between mass and weight is so important in Hydrostatical work that we propose to consider it here, together with the pound and poundal, although this more properly belongs to the study of Dynamics

Matter is a fundamental conception. We cannot define it satisfactorily, but we perceive it by our sense of touch and through the effort required to produce in it a sudden change of motion. This latter property we term *inertia* The measure of the inertia of a body is called its *mass*

The *weight* of a body is the force with which the earth attracts it and consequently, as with any other force, it has a *direction* associated with it The mass of a body has no direction, it is simply a quantity A distinction is made between quantities such as mass, temperature and volume (which have magnitude but no direction), these are called *scalar* quantities, and measures such as force, velocity and acceleration which have direction as well as magnitude, these are called *vectors* Thus mass is a scalar, but weight is a vector The unit of weight we have used so far is the weight of a one pound mass, and this has been convenient also as a unit of force It must be realised that a mass of one pound has a weight of one pound, because the earth attracts a one pound mass with a force of 1 lb wt Everywhere in this book we shall distinguish between units of mass and weight by writing lb for mass units and lb wt. for weight units Similarly in the C G.S system we shall distinguish grm. from grm. wt For this reason all the forces we used in Chapter I are given in lb wt , ton wt , or grm. wt

There is an alternative method of measuring forces to the lb wt which is fundamentally important The earth attracts a one pound mass with a slightly varying force according to its position on the earth Consequently 1 lb wt is not constant at all parts of the earth, and an alternative measurement of force is derived from Newton's Second Law of Motion. From this law it may be deduced (see *Tutorial Dynamics*, p 80) that if a force acts on a body at rest and produces an acceleration, then the force is proportional to the mass of the body multiplied by the acceleration Suppose the body to have a mass of

m lb. and the acceleration produced to be f ft. per sec. per sec., then if F (in unspecified units as yet) is the force, we have

$$F \propto mf,$$

or

$$F = kmf,$$

where k is the constant of proportionality. Since the unit of force is unspecified we may choose it to make $k = 1$, and then we may write

$$F \text{ units of force} = mf,$$

where m is in pounds and f in ft. per sec. per sec. Thus, when m is one pound and f is 1 ft. per sec. per sec. we have defined this new unit of force, *i.e.*

1 unit of force gives a mass of 1 lb., an acceleration of 1 ft./sec.^{2}.

It is this unit which is called a *poundal,* and since it does not depend on the slightly variable acceleration due to gravity it is termed an *absolute* unit whereas 1 lb. wt. is termed a *gravitational* unit of force. Since, however, both the poundal and the lb. wt. are units of force we may easily determine the relation between them. Assuming that at a certain place on the earth, g (the acceleration due to gravity) has the value 32 ft. per sec. per sec., then

1 lb. wt. gives a mass of 1 lb. an acceleration of 32 ft. per sec. per sec.,

but 1 poundal gives a mass of 1 lb. an acceleration of 1 ft. per sec. per sec. and consequently we require

32 poundals to give a mass of 1 lb. an acceleration of 32 ft. per sec. per sec.,

so that 32 poundals are equivalent to 1 lb. wt. at the particular place on the earth where $g = 32$ ft./sec.^{2}. At any other place, whatever is the numerical value of g, we shall have

g poundals equivalent to 1 lb. wt.

In C.G.S. units the *dyne* is the unit of force which corresponds to the poundal, and assuming g to be 981 cm./sec.^{2} we have

1 grm. wt. gives a mass of 1 grm. an acceleration of 981 cm. per sec. per sec.;

but 1 dyne gives a mass of 1 grm. an acceleration of 1 cm. per sec. per sec.,

so that

981 dynes give a mass of 1 grm. an acceleration of 981 cm. per sec. per sec.,

i.e. 981 dynes are equivalent to 1 grm. wt.

12. Density

The density of a substance is the mass of a unit volume. Thus it may be expressed in lb. per cubic foot, lb. per cubic inch, grammes per cubic centimetre and so on.

For example, a cubic foot of pure water at 4° C weighs very nearly 1000 ounces consequently 1 cub ft of pure water has a mass of 1000 oz, i e

$$\text{the density of pure water} = 1000 \text{ oz per cub ft}$$
$$= 62\ 5 \text{ lb per cub ft}$$

(In fact, 62 3 lb per cub ft is a more accurate value but 62 5, i e $\frac{1000}{16}$, is a more convenient arithmetic approximation)

In the C G S system one gramme is defined as the weight of one cubic centimetre of pure water at 4° C, so that the density of pure water = 1 grm per cub cm It is customary to denote the magnitude of the density of a substance by the Greek letter *rho* (ρ)

It is clear that if we have a body whose volume is V cub ft and whose density is ρ lb per cub ft, then its mass (m) in pounds is given by

$$m = V\rho$$

Example —*To find the density of lead, having given that a spherical bullet of lead 2 cm in diameter, weighs 45 7 grm*

The bullet is a sphere whose radius = 1 cm

Hence its volume $= \frac{4}{3} \times \frac{22}{7} \times (1)^3 = \frac{88}{21}$ cub cm

Also the mass of the bullet = 45 7 grm

density of lead $= 45\ 7 \times \frac{21}{88} = 10\ 9$ grm per cub cm

13 Specific gravity

The specific gravity of a substance is the ratio of the weight of any volume of that substance to the weight of an *equal* volume of the standard substance

The standard substance generally taken is pure water at a temperature of 4° C, and since a specific gravity is the ratio of one weight divided by another weight it is a pure number i e it has no dimensions For example if we find that a cubic foot of sea water weighs 64 lb we may find its specific gravity, remembering that a cubic foot of pure water weighs 62 5 lb

Thus, from the definition above,

$$\text{specific gravity of sea water} = \frac{64}{62\ 5} = 1\ 024$$

The abbreviation for specific gravity is sp gr

Again, the specific gravity of mercury is about 13 6 This implies that a cubic foot of mercury weighs 13 6 times as much as a cubic foot of pure water, i e it weighs

$$13\ 6 \times 1000 \text{ oz},$$
$$\text{or} \quad 850 \text{ lb}$$

We may easily establish the relation between the density and specific gravity of a substance as follows:—

$$\text{The sp. gr. of a substance} = \frac{\text{weight of any volume of substance}}{\text{weight of } \textit{equal} \text{ volume of water}} \quad \text{(by definition)}$$

$$= \frac{\text{mass of any volume of substance}}{\text{mass of equal volume of water}}$$

$$= \frac{\text{mass of } \textit{unit} \text{ volume of substance}}{\text{mass of } \textit{unit} \text{ volume of water}}$$

$$= \frac{\text{density of substance}}{\text{density of water}},$$

and consequently

$$\text{density of substance} = \text{sp. gr. of substance} \times \text{density of water.}$$

Notice also, that from the definition of specific gravity, the weight of any volume of substance = sp. gr. of substance × weight of equal volume of water, and this may conveniently be put in symbols. If a volume V cub. ft. of a substance of specific gravity s weighs W lb., and if the weight per unit volume of the standard substance (water) is w, then the weight of V cub. ft. of water is Vw and the above relation becomes

$$W = sVw.$$

In the C.G.S. system the gramme is chosen so that the density of pure water at 4° C. is 1 grm. per cub. cm., and then we have

$$\text{sp.gr. of substance} = \frac{\text{density of substance}}{\text{1 grm. per c.cm.}},$$

and thus the specific gravity of a substance is the same as the numerical value of its density in C.G.S. units.

Example 1.—*The density of a piece of crystal is* 155·75 *lb. per cubic foot. What is its specific gravity?*

Weight of a cubic foot of the crystal = 155·75 lb.

Weight of a cubic foot of water = 62·5 lb.;

$$\therefore \text{ Specific gravity} = \frac{\text{weight of substance}}{\text{weight of equal volume of water}}$$

$$= \frac{155 \cdot 75}{62 \cdot 5} = 2 \cdot 492.$$

Example 2.—*To find in ounces the weight of a cubic inch of lead taking the specific gravity of lead to be* 11·4.

Weight of a cubic foot of water = 1000 oz.;

weight of a cubic inch of water (i e of $\frac{1}{1728}$ cub ft) $= \frac{1000}{1728}$ oz

But a cubic inch of lead weighs 11 4 times as much,

$$\text{weight of a cubic inch of lead} = \frac{1000 \times 11\ 4}{1728} = 6\ 59 \text{ oz approx}$$

Example 3 —*If a cubic foot of water weighs 62 5 lb, what is the weight of 1 cub in of gold, the specific gravity of gold being 19 25 ?*

Using the notation of the last paragraph,

$w = 62\ 5$ lb wt, $s = 19\ 25$ and $V = 1$ cub in or $\frac{1}{1728}$ cub ft,

$$W = 19\ 25 \times \tfrac{1}{1728} \times 62\ 5 \text{ lb wt}$$
$$= 0\ 696 \text{ lb wt}$$

14 Specific gravity of mixtures

If given volumes of ingredients of known specific gravity are mixed we may calculate their separate weights and hence, knowing the total weight and the total volume, we may calculate the specific gravity of the mixture

Example 1 —*To find the specific gravity of a mixture of 2 cub ft of fresh water and 3 cub ft of sea water, having given that the specific gravity of sea-water is 1 026*

Here 2 cub ft of fresh water weigh 2000 oz and 3 cub ft of sea water weigh $3 \times 1\ 026 \times 1000$ oz $= 3078$ oz

Hence the weight of the mixture $= 5078$ oz

Also the volume of the mixture $= 5$ cub ft

the weight of an equal volume of water $= 5 \times 1000$ oz,

the specific gravity of the mixture $= \frac{5078}{5000} = 1\ 0156$

It is not really necessary to know the actual volumes of the components, provided that their relative proportions are known In this case, we may proceed as in the following examples, which may be taken as types

Example 2 —*To find the specific gravity of a mixture of 3 parts (by volume) alcohol, 2 parts water, and 1 part glycerine given that the specific gravity of alcohol is 0 794, and that of glycerine is 1 26*

Let w be the weight of 1 part of water

Then the weight of 3 parts of alcohol $= 3 \times 0\ 794w$

$= 2\ 382w$

Also the weight of 2 parts of water $= 2w$

Also the weight of 1 part of glycerine $= 1\ 26w$,

$\therefore$ the weight of 6 parts of the mixture $= 5{\cdot}642w$.

But the weight of an equal volume of water $= 6w$;

$$\therefore \text{ specific gravity of mixture} = \frac{5{\cdot}642}{6} = 0{\cdot}940\dot{3}.$$

Example 3.—*An amalgam is formed by mixing 3 volumes of potassium with 7 of mercury, the volume of the amalgam being four-fifths of that of its constituents. Find its specific gravity, being given that specific gravities of mercury and potassium are 13·596 and 0·860 respectively.*

Let w be the weight of 1 volume of water.

Then weight of potassium $= 3w \times 0{\cdot}860$,

weight of mercury $= 7w \times 13{\cdot}596$.

Volume of mercury and potassium $= 3$ vols. $+ 7$ vols. $= 10$ vols.,

and volume of amalgam $=$ four-fifths of this $= 8$ vols.;

$\therefore$ weight of equal volume of water $= 8w$;

$$\therefore \text{ specific gravity} = \frac{\text{weight of amalgam}}{\text{weight of equal volume of water}}$$

$$= \frac{3w \times 0{\cdot}860 + 7w \times 13{\cdot}596}{8w} = 12{\cdot}219.$$

(*a*) *Mixture by Volume.*—We may generalise the foregoing. If volumes $V_1, V_2, V_3, \ldots$ of fluids of specific gravities $s_1, s_2, s_3, \ldots$ are mixed together and the resulting volume U has a specific gravity S, then the weight of the mixture is

$$SUw,$$

where w is the weight per unit volume of the standard substance; and this total weight of the mixture must be equal to the sum of the constituent weights, *i.e.*

$$s_1V_1w + s_2V_2w + s_3V_3w + \ldots.$$

Thus $$SUw = s_1V_1w + s_2V_2w + s_3V_3w + \ldots$$

or $$S = (s_1V_1 + s_2V_2 + s_3V_3 + \ldots)/U.$$

If there is no contraction or expansion on mixing, then

$$U = V_1 + V_2 + V_3 + \ldots$$ and we have

$$S = \frac{s_1V_1 + s_2V_2 + s_3V_3 + \ldots}{V_1 + V_2 + V_3 + \ldots}.$$

(*b*) *Mixture by Weight.*—If weights $W_1, W_2, W_3, \ldots$ of the fluids of specific gravity $s_1, s_2, s_3 \ldots$ are mixed together and the resulting mixture has a *volume* U and specific gravity S; then as before the weight of the mixture is

$$SUw,$$

where w is the weight per unit volume of the standard substance.

Since the total weight is the sum of the weights of the constituents

$$SUw = W_1 + W_2 + W_3 +$$

so that $$S = (W_1 + W_2 + W_3 + \quad)/Uw$$

If there is no contraction or expansion on mixing then U, the final volume, is the sum of the volumes of the constituents, and these are $\frac{W_1}{ws_1}$, $\frac{W_2}{ws_2}$, $\frac{W_3}{ws_3}$,

so that $$U = \frac{W_1}{ws_1} + \frac{W_2}{ws_2} + \frac{W_3}{ws_3} +$$

and then $$S = \frac{W_1 + W_2 + W_3 +}{\frac{W_1}{s_1} + \frac{W_2}{s_2} + \frac{W_3}{s_3} +}$$

We may conveniently express the results we have obtained in the *sigma* (Σ) notation, where we write

$$\Sigma W_r \text{ for } W_1 + W_2 + W_3 +$$

the $\Sigma W r$ simply indicating the sum of all expressions such as W_1, W_2 etc Similarly we may write

$$\Sigma(s_r V_r) \text{ for } s_1 V_1 + s_2 V_2 + s_3 V_3 +$$

and so for the mixture by volume we have

$$S = \frac{1}{U}\Sigma(s_r V_r) \text{ where the final volume is U,}$$

or $$S = \frac{\Sigma(s_r V_r)}{\Sigma V_r} \text{ if there is no change in volume}$$

Again, for the mixture by weight

$$S = \frac{1}{Uw}\Sigma W_r,$$

or $$S = \frac{\Sigma W_r}{\Sigma\left(\frac{W_r}{s_r}\right)} \text{ if there is no change in volume}$$

Example 1.—*To find the weights of copper (specific gravity 8 8) and zinc (specific gravity 7) in 1 lb of brass (specific gravity 8)*

Let the weight of copper be x lb

Then the weight of zinc is $1 - x$ lb

Thus the *volume* of copper is $\frac{x}{8\,8w}$ cub ft where w is the weight in lb per cubic foot of the standard (water) Also the volume of zinc is $\frac{1-x}{7w}$

and the combined volume of brass is $\frac{1}{8w}$.

Since we assume that there is no change in volume

$$\text{Vol. copper} + \text{vol. zinc} = \text{vol. brass},$$

i.e.
$$\frac{x}{8{\cdot}8w} + \frac{1-x}{7w} = \frac{1}{8w}$$

or
$$7x + 8{\cdot}8\,(1-x) = \frac{7 \times 8{\cdot}8}{8}$$

$$= 7{\cdot}7;$$

$$\therefore \quad 8{\cdot}8 - 7{\cdot}7 = 8{\cdot}8x - 7x$$

or
$$1{\cdot}8x = 1{\cdot}1,$$

i.e.
$$x = \tfrac{11}{18} \text{ lb. wt.}$$

Thus in 1 lb. brass there is $\frac{11}{18}$ lb. of copper and $\frac{7}{18}$ lb. of zinc.

Example 2.—*A mixture has to be made by taking m parts by weight of one substance and n parts by weight of another. Instead of this, m parts by volume of the first and n parts by volume of the second are taken. Show that the specific gravity of the mixture is greater than if the proper proportions were taken.*

Let the specific gravities of the two substances be s_1 and s_2 and let S_w, S_v be the specific gravities of the mixtures when taken correctly by weight, and incorrectly by volume respectively.

Suppose the weights of the ingredients which should be taken are m lb. of the substance with specific gravity s_1, and n lb. of the substance with specific gravity s_2. Then

$$S_w = \frac{m+n}{\dfrac{m}{s_1} + \dfrac{n}{s_2}}.$$

Suppose, also, that the volumes of the ingredients which are actually taken are m cub. ft. and n cub. ft.

Then
$$S_v = \frac{ms_1 + ns_2}{m+n}.$$

It is required to show that $S_v > S_w$,

i.e. that
$$\frac{ms_1 + ns_2}{m+n} > \frac{m+n}{\dfrac{m}{s_1} + \dfrac{n}{s_2}},$$

i.e. that

$$m^2 + mn\frac{s_1}{s_2} + mn\frac{s_2}{s_1} + n^2 > m^2 + 2mn + n^2 \text{ by cross-multiplying,}$$

i.e. that
$$mn\left(\frac{s_1}{s_2} + \frac{s_2}{s_1}\right) > 2mn,$$

$$\text{i e that} \qquad \frac{s_1^2 + s_2^2}{s_1 s_2} > 2,$$

$$\text{i e that} \qquad s_1^2 + s_2^2 > 2s_1 s_2,$$

$$\text{i e that} \qquad s_1^2 - 2s_1 s_2 + s_2^2 > 0,$$

which it is, because the left hand side is $(s_1 - s_2)^2$ which is always a positive quantity and therefore greater than zero Hence $S_v > S_w$

Exercises II

(1) If the rainfall is 1 in, how many tons of water fall on an acre?

(2) A tank weighs 1 ton empty and 2 50 ton when filled with pure water. If the tank weighs 2 95 ton when filled with oil, find the specific gravity of the oil

(3) The weight of a lorry and empty milk container (capacity 4 cubic yards) is 3 ton When filled with milk, the lorry and container weighs 6 10 ton Find the specific gravity of the milk

(4) If 5 cub in of mercury weigh 2 45 lb and 2 cub in of cast iron weigh 0 52 lb, what ratio does the density of mercury bear to that of cast iron?

(5) The density of cast iron in the C G S system of units is 7 2 What is its density in the foot pound system of units?

(6) The outer radius of a spherical leaden bullet containing a spherical cavity is R, and its weight is W If w is the weight of a unit volume of lead, show that the radius of the cavity

$$= \sqrt[3]{\left\{R^3 - \frac{3}{4\pi}\frac{W}{w}\right\}}$$

(7) A body has a volume of 54 cub ft and specific gravity of 1 1 A second body has a volume of $\frac{1}{8}$ cub ft and specific gravity 4 95 What is the ratio of the mass of the first body to that of the second?

(8) Find the specific gravity of a mixture of 2000 oz of fresh water with 3000 oz of sea water (sp gr sea water is 1 026)

(9) Find the specific gravity of a mixture of 3 parts by weight of alcohol (sp gr 0 784), 2 parts by weight of water and 1 part by weight of glycerine (sp gr 1 26)

(10) A nugget of gold mixed with quartz weighs 12 oz, and has a specific gravity of 6 4, given that the specific gravity of gold is 19 35, and of quartz is 2 15, find (to one place of decimals) the quantity of gold in the nugget

(11) Four pints of alcohol, having a specific gravity of 0 75, are mixed with one pint of water (specific gravity 1) Find the specific gravity of the mixture, no change of volume being supposed to take place

(12) Three equal vessels A, B, C are half-full of liquids, densities d_1, d_2, d_3, respectively. If now B is filled up from A and then C from B, find the density of the liquid now contained in C, the liquids being supposed to mix completely.

(13) A substance whose specific gravity is 0·7 is dissolved in 10 times its own weight of water, and the specific gravity of the solution is 1·01. Find by how much the total volume is reduced.

(14) If a volume v_1 of a liquid whose specific gravity is s_1 be mixed with a volume v_2 of a liquid whose specific gravity is s_2, and the specific gravity of the mixture is s, find the change in volume.

(15) Two liquids A and B have densities in the ratio of 4 : 5. When certain volumes are mixed, the volume of the mixture is found to be one-eighteenth part less than the sum of the separate volumes, and the density of the mixture is to that of A as 9 : 8. Find the ratio of the volume of A to the volume of B.

ANSWERS

1. 101·28. 2. 1·3. 3. 1·029.

4. 49 : 26, or 1·88 : 1. 5. 450. 7. 96 : 1.

8. 1·0154. 9. 0·913. 10. 8·96.

11. 0·8. 12. $(d_1 + d_2 + 2d_3)/4$.

13. 4·703% of the original volume.

14. $\{v_1(s_1 - s) + v_2(s_2 - s)\}/s$. 15. 3 : 1.

CHAPTER III

PRESSURE OF HEAVY FLUIDS (1)

Pressure Intensity in Fluids

15 Equal pressure intensity at all points in a horizontal plane

In a fluid at rest under gravity, the pressure intensity at all points in the same horizontal plane is the same

This is the first of a number of easily established theorems on the pressure intensities at points within a fluid at rest due to the weight of the fluid

Let A and B be two points in a fluid in the same horizontal plane (Fig 9)

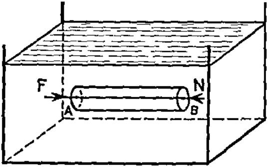


Fig 9

Join AB and imagine a cylinder constructed in the fluid with AB as axis, of cross sectional area a sq in Then the equilibrium of this cylinder of fluid depends on four forces —

(1) The weight of the fluid in the cylinder, acting vertically downwards

(2) The force exerted on the cylinder of fluid by the external fluid across the curved surface of the cylinder This will be everywhere normal to the curved surface and consequently in the vertical plane

(3) The force F lb wt exerted on the fluid in the cylinder across the end at A

(4) A similar force N lb wt across the end at B

Consequently if we resolve all the forces horizontally we have only to consider the last two and we have

$$F = N$$

Thus the average pressure over the end at A is $\frac{F}{a}$ lb. wt. per sq. in., and this is equal to the average pressure $\left(\frac{N}{a}\text{ lb. wt. per sq. in.}\right)$ over the end at B.

If we now imagine the cylinder to shrink, both F (or N) and a diminish, and we have

limiting value, as a tends to zero, of $\frac{F}{a}$

$$= \text{limiting value, as } a \text{ tends to zero, of } \frac{N}{a},$$

or $$\lim_{a \to 0} \frac{F}{a} = \lim_{a \to 0} \frac{N}{a}$$

as it is written in mathematical work

$$\left(\text{or alternatively, } \quad \underset{a \to 0}{\mathrm{L}} \frac{F}{a} = \underset{a \to 0}{\mathrm{L}} \frac{N}{a}\right).$$

Now this limiting value was defined as the pressure intensity of the point (§ 6) thus

pressure intensity at A = pressure intensity at B.

We have shown, so far, that the pressure intensities at points in the same horizontal *line* are equal, but since the pressure intensity at any point in a fluid is the same in all directions (§ 7) we deduce that the pressure intensities at all points in the same horizontal *plane* are equal.

A *homogeneous* fluid is defined as one in which, if any equal volumes are taken, the masses of these equal volumes are equal, *i.e.* the density is everywhere constant.

The theorem we have just proved is true whether the fluid is homogeneous or not, since the density was not involved.

16. In homogeneous fluid at rest under gravity the difference in pressure intensity of any two points is proportional to the difference in their depths

Let A and B be two points in the same vertical line and suppose that a cylinder is constructed about AB as axis, having a cross-sectional area of a sq. in. (Fig. 10). Let p_A, p_B be the pressure intensities in lb. wt. per sq. in. at A and B respectively, and suppose that the fluid has a density of ρ lb. per cub. in.

Consider the equilibrium of the cylinder of fluid; it is acted upon by the following forces:—

(1) Its weight, W lb. wt.

(2) The force due to the external fluid on the end at A, this is $p_A a$ lb wt

(3) The force due to the external fluid on the end at B, this is $p_B a$ lb wt

(4) The force due to external fluid on the curved surface of the cylinder This acts in the horizontal plane

If we resolve all the forces vertically we shall have a relation between the first three of those enumerated above—the fourth will not here concern us

Thus (Fig 11),

$$p_A a + W = p_B a$$

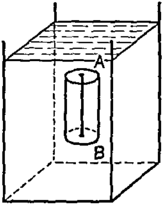


Fig 10

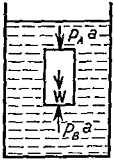


Fig 11

But if AB = z in, then the volume of the cylinder is za cub in and consequently its mass is ρza lb Thus its weight is ρza lb wt, and the relation above becomes

$$p_B a = p_A a + \rho z a$$

or

$$p_B - p_A = \rho z$$

or

$$p_B - p_A \propto z,$$

and since z is the difference in depth of the two points A and B, the theorem is proved for the case where the points are in the same vertical line, and we may extend this result to the case where the two points are not in the same vertical

Consider Fig 12 If N is the point vertically below A and in the same horizontal as B, then by the last theorem, using the same notation, but with z = AN

$$p_N - p_A = \rho z$$

But from § 15, $p_N = p_B$, because the pressure intensity at points in the same horizontal line is the same. Thus

$$p_B - p_A = \rho z$$

as before.

Now suppose the horizontal through B does not lie wholly in the fluid (Fig. 13), but suppose we may draw a horizontal line NM where N and M are on the verticals through A and B respectively.

Then, with notation as before, it is clear that

$$p_N - p_A = \rho . AN \quad \text{(i)}$$

$$p_N = p_M \quad \text{(ii)}$$

$$p_M - p_B = \rho . BM \quad \text{(iii)}$$

Subtracting (iii) from (i)

$$p_N - p_A - p_M + p_B = \rho . AN - \rho . BM;$$

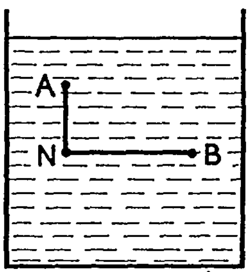


Fig. 12.

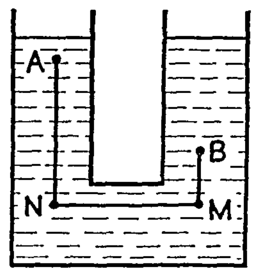


Fig. 13.

$\therefore$ Using (ii) $$p_B - p_A = \rho\,(AN - BM)$$
$$= \rho \times \text{difference in depths of A and B.}$$

Even this does not exhaust all the possibilities, but we may always proceed by a series of steps when the same result will be found valid. Consider Fig. 14, and the steps that have been drawn. The proof in this case is similar to the last, the relations are clearly

$$p_N - p_A = \rho . AN \quad \text{(i)}$$

$$p_N = p_M \quad \text{(ii)}$$

$$p_M - p_L = \rho . LM \quad \text{(iii)}$$

$$p_L = p_S \quad \text{(iv)}$$

$$p_S - p_T = \rho . TS \quad \text{(v)}$$

and $$p_T = p_B \quad \text{(vi)}$$

Add (iii) to (v).

$$p_M - p_L + p_S - p_T = \rho\,(LM + TS).$$

Relation (iv) enables us to cancel $-p_L$ with $+p_S$ then, using (ii) and (vi) this becomes

$$p_A - p_B = \rho\,(LM + TS)$$

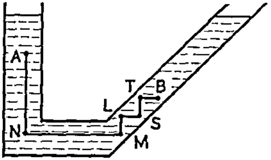


Fig 14

Subtracting this from (i)

$$p_A - p_A - p_A + p_B = \rho\,(AN - LM - TS),$$

or

$$p_B - p_A = \rho\,(AN - LM - TS)$$

$$= \rho \times \text{difference in depths of A and B}$$

17. To find the pressure intensity at any depth of a heavy homogeneous liquid

In the last section we showed that the difference of pressure intensity of two points was proportional to the difference in their depths, and we may use that result to determine the actual pressure intensity at any given depth Using the notation of the last section but taking A in the surface and as before AB = z, we have

$$p_B - \text{pressure intensity at surface} = \rho z$$

Now the pressure intensity at the surface *is not zero* as might be imagined The earth's atmosphere produces a pressure, known as *atmospheric pressure*, which acts on all surfaces and consequently on the free surface of a liquid This atmospheric pressure varies continuously, but it is of the order of 14 7 lb wt per sq in We shall discuss experimental methods of determining atmospheric pressure in Chapter X

This atmospheric pressure is then the pressure intensity at the surface of the liquid—we shall denote it by P lb wt per sq in —when the last equation becomes

$$p_B - P = \rho z$$

or

$$p_B = P + \rho z$$

Example 2.—*At what depth in the sea (specific gravity 1·024) is the pressure intensity double the atmospheric pressure (14·7 lb. wt. per sq. in.)?*

Let the depth be l ft., then

pressure intensity at depth of l ft. $= 14·7$ lb. wt./sq. in. $+ \rho l$

This is twice atmospheric pressure when

$$\rho l = 14·7 \text{ lb. wt./sq. in.}$$

But ρ (the density of sea water) $= 1·024 \times 62·5$ lb. per cub. ft.

$$\rho l = 1·024 \times 62·5 \times l \text{ lb. per sq. ft.}$$

$$= \frac{1·024 \times 62·5 \times l}{144} \text{ lb. per sq. in.}$$

Thus $\dfrac{1·024 \times 62·5 \times l}{144} = 14·7$

giving $l = 33·75$ ft.

Example 3.—*A corked-up bottle is lowered to a depth of 28 ft. in water and the cork is $\frac{1}{16}$ ft. in diameter. What is the force tending to drive the cork in?*

The pressure intensity at depth of 28 ft. $= 28\,000$ oz. wt. per sq. ft.

Also the diameter of the cork $= \frac{1}{16}$ ft.

its area $= \frac{22}{7} \times (\frac{1}{32})^2$ sq. ft.

and the force on the cork

$= \frac{22}{7} \times \frac{1}{32} \times \frac{1}{32} \times 28\,000$ oz. wt. $= \frac{1375}{16}$ oz. wt. $= 85\frac{15}{16}$ oz. wt.

$= 5$ lb. $5\frac{15}{16}$ oz. wt.

18. The free surface of a heavy liquid at rest is horizontal.

Let M and N be two points in a horizontal plane in a liquid vertically below any points A and B in the surface, and let p_M and p_N be the pressure intensities at M and N respectively (Fig. 16).

Then from § 15 $p_M = p_N$

and from § 17 $p_M = P + \rho \cdot AM$

and $p_N = P + \rho \cdot BN$

Thus $P + \rho \cdot AM = P + \rho \cdot BN$

i.e. $AM = BN$

and so AB is parallel to MN

AB is horizontal.

But A and B were any two points in the surface, therefore the surface is horizontal.

19. The surface of a liquid at rest rises everywhere to the same level

It is important to notice that we did not assume this in § 16. There, we were concerned with the difference in pressure intensity at points in a liquid, and we showed that this difference was proportional to the vertical distance between one point and a horizontal line through the other point. Consequently this did not assume what we are now considering; namely, that the surface of a liquid at rest rises everywhere to the same level.

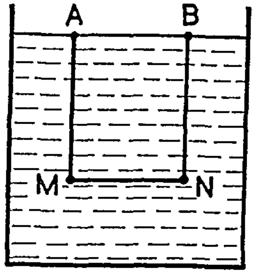


Fig. 16.

Experimental Illustration.—This property may be verified experimentally by constructing an apparatus such as that shown in Fig. 17, in which several open vessels of different shapes and sizes D, E, F communicate freely with one another. If water or any other liquid be poured into one of them, it will rise to the same level in them all.

The proof of § 18 holds when the liquid is contained in two or more communicating vessels such as D, E. In this case all the free surfaces are at the same level, and form part of one and the same horizontal plane. If the liquid is contained in a vessel such as that shown at F the proof fails, for we cannot construct a vertical column whose base is at a point R of the bottom without passing out of the liquid.

But we can always connect any point R with the surface by means

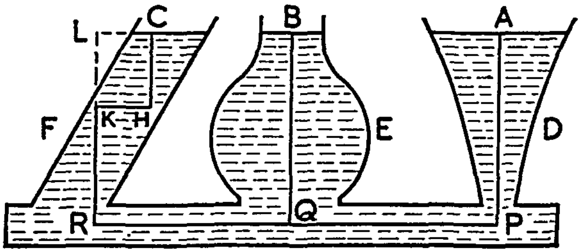


Fig. 17.

of a zigzag of alternately vertical and horizontal straight lines CH, HK, KR, and can find the difference between the pressure intensities at any two points on this zigzag as we did in § 16.

Thus, since CH is vertical, we have

$$\text{pressure intensity at H} = \rho \times \text{CH} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where ρ is the weight of unit volume of liquid.

Since KH is horizontal,

pressure intensity at K − pressure intensity at H = 0 (ii)

Since KR is vertical,

pressure intensity at R − pressure intensity at K = $p \times$ KR (iii)

Therefore, by adding (i) (ii), (iii),

$$\text{pressure intensity at R} = \rho \times (\text{CH} + \text{KR})$$
$$= \rho \times \text{depth of R below surface}$$

We may now show that the liquid in the vessel F reaches the same level as in D For, if R is on the same level as P, the pressures at R P are equal, and therefore the depth of R below the surface at C is equal to that of P below the surface at A Hence AC is horizontal

20 To find the pressure intensity at any depth in a liquid which has other liquids superimposed without mixing

We may adopt an exactly similar method to § 16 by examining the equilibrium of a column of liquid Suppose we have three liquids of densities ρ_1 ρ_2 ρ_3 lb per cub in as shown in Fig 18 Consider the equilibrium of a vertical cylinder of liquid (AB), of cross sectional area a sq in As before by resolving all the forces vertically we have

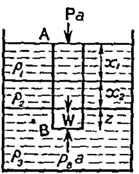


Fig 18

$$p_B a = \text{P}a + \text{W} \quad \text{(i)}$$

where p_B is the pressure intensity at B in lb wt per sq in P is the atmospheric pressure also in lb wt per sq in and W is the total weight of the cylinder of liquid in lb wt If the thicknesses of the top two layers of liquid are x_1 and x_2 in and if the point B is z in below the surface of the lowest liquid, then

$$\text{W} = ax_1\rho_1 + ax_2\rho_2 + az\rho_3 \quad \text{(ii)}$$

since the volume of that part of the cylinder in the uppermost liquid is ax_1 cub in and consequently its mass is $ax_1\rho_1$ lb, and thus its weight is $ax_1\rho_1$ lb wt, and similarly for the remaining parts of the cylinder

Substituting the result of (ii) in (i) we have

$$p_B a = \text{P}a + a(x_1\rho_1 + x_2\rho_2 + z\rho_3)$$

or

$$p_B = \text{P} + x_1\rho_1 + x_2\rho_2 + z\rho_3$$

giving the pressure intensity at the point B

If we have any number of liquids of densities ρ_1, ρ_2 ρ_3 lb per cub in, and if their respective thicknesses are x_1 x_2 x_3 in, then

the pressure intensity at the lowest point (B) of the lowest liquid is clearly given by

$$p_B = P + \Sigma x_r \rho_r.$$

21. To find the pressure intensity at any depth of a heavy liquid which is not homogeneous

This is a very simple and instructive extension of the last section for those students who have a little knowledge of the Calculus, if we know how the density varies with the depth. Consider a vertical cylinder of liquid, cross-sectional area a sq. in., and suppose we examine the equilibrium of a small element of this cylinder (Fig. 19) of thickness δz and at a depth of z in.

Let p lb. wt. per sq. in. be the pressure intensity at the depth z and $p + \delta p$ the pressure intensity at the depth $z + \delta z$. The weight of the element is $\rho . a\ \delta z$, and consequently by resolving the forces vertically as before, we have

$$(p + \delta p)\, a = pa + pa\delta z,$$

or

$$\delta p = \rho \delta z;$$

$$\therefore \quad \frac{\delta p}{\delta z} = \rho,$$

and thus in the limiting case

$$\frac{dp}{dz} = \rho,$$

so that

$$p = \int \rho dz + a \text{ constant.}$$

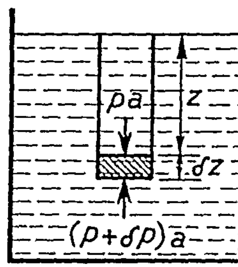


Fig. 19.

The constant is the value of the pressure intensity when $z = 0$, *i.e.* at the free surface, and this is P, the atmospheric pressure.

Thus, $$p = P + \int \rho dz,$$

where the integration extends from zero up to the value of z at which the pressure intensity is required.

Example 1.—*To determine the pressure intensity at a depth of 1 ft. in a liquid whose density at a point z in. below its surface is $\frac{1}{4}z$ lb. per cub. in.*

From the above, if p is the pressure intensity required in lb. wt. per sq. in., then

$$p = P + \int_0^{12} \rho dz,$$

i.e.

$$p = P + \int_0^{12} \tfrac{1}{4} z dz$$

$$= P + \left[\frac{1}{4} \cdot \frac{z^2}{2}\right]_0^{12}$$

$$= P + \frac{144}{8}.$$

and taking P *as* 14 7 lb wt per sq in

$$p = (14\ 7 + 18)\ \text{lb wt per sq in}$$
$$= 32\ 7\ \text{lb wt per sq in}$$

22 If two heavy homogeneous liquids do not mix, their common surface is a horizontal plane

Let M N be two points in a horizontal line in the lower liquid, vertically below points A, B in the surface respectively and C, D in the common surface respectively (Fig 20)

Then from § 15,

$$\text{pressure intensity at M} = \text{pressure intensity at N}$$

i e (from § 20)

$$P + \rho_1\ AC + \rho_2\ CM = P + \rho_1\ BD + \rho_2\ DN$$

Putting CM = AM − AC and DN = BN − BD this is

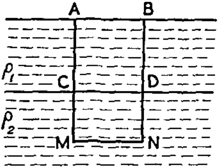


Fig 20

$$\rho_1\ AC + \rho_2\,(AM - AC) = \rho_1\ BD + \rho_2\,(BN - BD)$$

i e $$(\rho_1 - \rho_2)\ AC + \rho_2\ AM = (\rho_1 - \rho_2)\ BD + \rho_2\ BN \qquad (1)$$

Now MN is horizontal (given) and AB is horizontal (§ 18)

$$AM = BN$$

(1) becomes $$(\rho_1 - \rho_2)\ AC = (\rho_1 - \rho_2)\ BD$$

Either $$\rho_1 = \rho_2 \text{ or } AC = BD$$

Since the liquids have different densities ρ_1 is not equal to ρ_2

$$AC = BD$$

CD is parallel to AB

CD is horizontal

Example 1—*A glass tube of uniform bore and open at both ends is bent into the form of a letter* U *(known as a* U *tube) and contains pure*

water. Into one branch a liquid of specific gravity s is poured until there is a length l ft., while into the other branch a liquid of specific gravity S is poured until the levels of the two liquids in the two branches are the same. To find the length (L ft.) of liquid in the second branch.

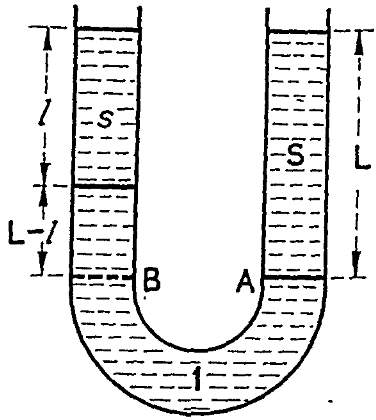


Fig. 21.

Consider Fig. 21 and let A be a point in the common surface of the water and liquid of specific gravity S. Then if B is any point in the same horizontal as A, but in the other branch of the tube:

pressure intensity at A = pressure intensity at B.

The pressure intensity at A, due to the liquid alone, is

$$L \times S \times 62\tfrac{1}{2} \text{ lb. wt./sq. ft.},$$

and at B it is

$$l \times s \times 62\tfrac{1}{2} + (L - l) \times 1 \times 62\tfrac{1}{2} \text{ lb. wt./sq. ft.};$$

$$\therefore \quad LS = ls + (L - l);$$

$$\therefore \quad l(1 - s) = L(1 - S);$$

$$\therefore \quad L = \frac{1 - s}{1 - S}\, l \text{ ft.}$$

Example 2.—*A narrow circular tube of uniform cross-section is made in the form of a semicircle which is mounted in a vertical plane, the open ends being uppermost. The tube contains columns of two liquids which do not mix, the density of one being twice that of the other. If the denser liquid subtends an angle of 90° at the centre, and the other an angle of 45°, find the angle which the radius through the common surface makes with the vertical.*

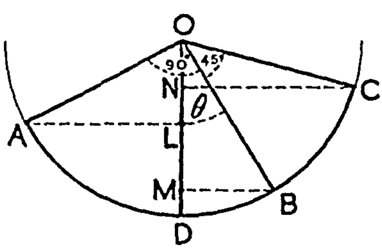


Fig. 22.

Let the density of one liquid be ρ lb. per cub. ft. and then the density of the other liquid is 2ρ lb. per cub. ft. Consider Fig. 22.

Let AB, BC represent the columns of liquids of densities 2ρ, ρ respectively, and let D be the lowest point of the semicircle. Join D to the centre of the circle

(O) and draw perpendiculars from A, B, C on to OD in L, M, N respectively Let θ be the angle required, i e, the $\angle$BOD Write LD $= x$, MD $= y$, ND $= z$ The pressure intensity at D due to the liquid AD is 2ρ LD or $2\rho x$ Also the pressure intensity at D due to the liquids BC and BD is

$$2\rho \text{ MD} + \rho \text{ NM}$$
$$= 2\rho y + \rho (z - y)$$

Since these pressure intensities are equal,

$$2\rho x = 2\rho y + \rho (z - y),$$

or

$$2x = 2y + z - y,$$

or

$$2x = y + z \qquad (i)$$

From the geometry of the figure, if r is the radius of the circle

$$x = \text{LD} = \text{OD} - \text{OL}$$
$$= r - r \cos (90 - \theta)$$
$$= r - r \sin \theta,$$

and

$$y = \text{MD} = \text{OD} - \text{OM}$$
$$= r - r \cos \theta$$

and

$$z = \text{ND} = \text{OD} - \text{ON}$$
$$= r - r \cos (45 + \theta)$$

Substituting these values of x, y and z in (i)

$$2r - 2r \sin \theta = r - r \cos \theta + r - r \cos (45 + \theta),$$
$$2 \sin \theta - \cos \theta = \cos (45 + \theta)$$
$$= \cos \theta \cos 45 - \sin \theta \sin 45$$
$$= \frac{1}{\sqrt{2}} (\cos \theta - \sin \theta)$$
$$2\sqrt{2} \sin \theta - \sqrt{2} \cos \theta = \cos \theta - \sin \theta,$$

i e

$$\sin \theta (1 + 2\sqrt{2}) = \cos \theta (1 + \sqrt{2}),$$
$$\tan \theta = \frac{1 + \sqrt{2}}{1 + 2\sqrt{2}} = 0\ 63$$

so that

$$\theta = 32^\circ\ 13$$

Exercises III

(1) A long glass tube of 1 in diameter has a disc weighing 2 oz placed at one end How far under water must the end of the tube, with the disc below it, be immersed in order that the disc may not fall off

(2) Determine the greatest depth in fathoms at which a submarine diver can work in sea water, supposing he can bear a pressure of 5 atmospheres, taking an atmosphere to be a pressure of 15 lb per sq in (1 cub ft of sea water weighs 64 lb)

(3) A hole 6 in. square is made in a ship's bottom 20 ft. below the water-line. What force must be exerted in order to keep the water out, by holding a piece of wood against the hole, if a cubic foot of water weighs 64 lb.?

(4) Find the pressure at a given depth (z) in a liquid whose specific gravity is s, and whose surface is subject to a given pressure P.

(5) Show that the effect of an external pressure of $13\frac{8}{9}$ lb. wt. per sq. in. may be allowed for when the liquid is water, by supposing a layer of water, 32 ft. thick, to be superposed on the original liquid.

(6) The pressure intensity at a point 3 ft. below the surface of a heavy fluid is 30 lb. wt. per sq. in., and at a depth of 7 ft. it is 50 lb. wt. What is the pressure intensity at the surface?

(7) A vessel whose bottom is horizontal contains mercury whose depth is 20 in., and water floats on the mercury to the depth of 16 in. Find the pressure intensity at a point on the bottom of the vessel in lb. wt. per sq. in., specific gravity of mercury being 13·6. Neglect atmospheric pressure.

(8) A circular cylinder, whose radius is 14 cm. and height 40 cm., is filled half with water and half with oil of specific gravity 0·9. Find the thrust on the base. (Take $\pi = \frac{22}{7}$.)

(9) The two branches of a uniform bent tube are straight and vertical, and the portion of the tube which unites them is horizontal. Water is poured in sufficient to fill 6 in. of the tube, and then oil, sufficient to occupy 5 in., is poured in at one end, the specific gravity of the oil being four-fifths that of water. Find the position of the fluids when they are in equilibrium, the horizontal part of the tube being 2 in. long.

(10) A U-tube contains mercury of specific gravity 13·5 to within 5 in. of the top. Find the height of the column of water which must be poured in to fill one of the branches.

(11) Water is poured into a U-tube, the legs of which are 8 in. long, until they are half full. As much oil as possible is then poured into one of the legs. What length of the tube does it occupy, the weight of the oil being two-thirds that of water?

(12) A bent tube containing equal quantities of two liquids which do not mix consists of two branches inclined at angle 60°. When one of the branches is held vertically, the different fluids meet at the angle of the tube. Show that when the tube is held with the two branches equally inclined to the vertical, one-fourth of the liquid contained in the branch which was previously inclined to the vertical flows into the one which was vertical.

(13) A tank has n layers of liquid each h inches thick, the uppermost having a density ρ lb. per cub. in., that of the next 2ρ, then 3ρ, and so on. Find the pressure intensity at the base, due to the liquids alone.

(14) Ten liquids (of densities ρ 2ρ, 3ρ, 10ρ) are superimposed, the thickness of the uppermost layer being h inches, that of the next $2h$ inches, then $3h$ inches and so on, so that the thickness of the lowest layer is $10h$ in Find the pressure intensity, due to the liquids alone at the lowest point of the lowest liquid

(15) In a heterogeneous fluid the density at a depth z inches is kz lb per cub in Find k if the pressure intensity at a depth of 2 ft is 40 lb wt per sq in Take the atmospheric pressure as 14 7 lb wt per sq in

(16) A thin circular tube is fixed with its plane vertical and contains two liquids which do not mix Each liquid subtends a right angle at the centre of the circle but they are of different densities ρ_1, ρ_2 $(\rho_1 < \rho_2)$ Find the angle which the radius to their junction makes with the vertical (Inter Eng)

(17) A circular tube, held in a vertical plane, contains columns of two liquids whose densities are ρ, σ, the columns subtending angles 2θ, 2ϕ respectively at the centre of the circle Show that the angle α, made by the diameter through the common surface with the vertical, is given by

$$\rho \sin \theta \sin (\theta \pm \alpha) = \sigma \sin \phi \sin (\phi \mp \alpha)$$

In particular, find α if equal volumes of the two liquids exactly fill half the tube

ANSWERS

1 4 4 in 2 $22\frac{1}{2}$ fathoms 3 320 lb wt

4 P + zsw where w is the weight of unit volume of standard substance.

6 15 lb wt per sq in 7 $10\frac{5}{12}$ lb wt per sq in 8 23 408 kilog

9 Common surface is 4 in below water surface and 5 in below oil surface

10 $5\frac{5}{26}$ in 11 6 in

13 $55\rho h$ lb wt per sq in 14 $385\ \rho h$ lb wt per sq in

15 0 088 16 $\tan^{-1}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)$ 17 $\tan^{-1}\left(\frac{\rho - \sigma}{\rho + \sigma}\right)$

CHAPTER IV

PRESSURE OF HEAVY FLUIDS (2)

TOTAL THRUST ON PLANE AREAS

23. Introduction

In the last chapter we dealt with the pressure intensity at various points within a fluid or series of fluids which did not mix. We considered various theorems concerning the pressure intensity at different points and we can now use the knowledge gained to find the total pressure or total thrust, as it is often termed, on a plane area immersed in a fluid. This is, obviously, of very great practical importance.

We know already (§ 4) that when a fluid is in contact with a plane area every part of the fluid exerts a force on the area, at right angles to it. It is the sum of all these parallel forces which we denote by the total thrust. It is often convenient to denote the sum of all these parallel forces by a single force—the resultant thrust—whose magnitude is, of course, the total thrust or total pressure on the area. Consequently we know the direction of this resultant thrust, it is normal to the area, because the resultant of a number of parallel forces is itself parallel to them. The resultant thrust will not be completely specified until we know the position where it acts on the area. This is an important point, called the *Centre of Pressure*, and we leave the determination of this point, in various cases, to Chapter V. In this chapter we will concern ourselves simply with the magnitude of the total thrust and we begin by taking the simplest case.

24. Total thrust on horizontal plane areas

In § 15 we found that the pressure intensity on a fluid at all points in a horizontal plane was constant and so the total thrust in this case is the pressure intensity at the depth of the plane figure multiplied by its area.

Example 1.—*A rectangular tank whose base measures 12 ft. × 4 ft. has a liquid (specific gravity 1·9) to a depth of 5 ft. To find the total liquid pressure on the base.*

The pressure intensity at the base due to the liquid

$$= 1{\cdot}9 \times 62{\cdot}5 \times 5 \text{ lb. wt. per sq. ft.}$$

The area of the base is 4×12 sq ft Thus, the total thrust on base

$$= 4 \times 12 \times 1\ 9 \times 62\ 5 \times 5 \text{ lb wt}$$

$$= \frac{4 \times 12 \times 1\ 9 \times 62\ 5 \times 5}{2240} \text{ ton wt}$$

$$= 127\ 23 \text{ ton wt}$$

It will be noticed here and in the following cases that we are only concerned with the thrust due to the weight of the liquid and so we shall leave out atmospheric pressure

25 The total thrust exerted by a liquid of given depth on the base of its containing vessel is independent of the shape of the remaining portion of the vessel

This follows from § 17, for the pressure intensity at any point of the base depends only on the depth of the liquid and the density,

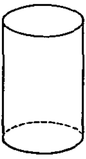


Fig 23

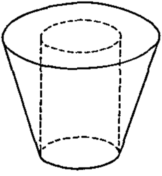


Fig 24

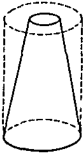


Fig 25

and not on the shape of the other part of the containing vessel, and the total thrust on the base depends only on this pressure intensity and the area of the base

Although this follows directly from what has gone before, nevertheless, it requires a closer inspection Consider the following cases —

(1) When liquid is contained in an open cylindrical vessel (Fig 23), the liquid exercises a thrust on the base which is vertically downward, and thrusts on the curved surface which are horizontal and consequently, when we resolve vertically,

total thrust on base = weight of cylinder of fluid

(2) Suppose the vessel is "pail" shaped (Fig 24), then the liquid exerts a thrust on the base and also thrusts on the curved surface, at right angles to the curved surface, and which consequently have a vertical downward component Thus on resolving vertically we have

thrust on base + resolved part downwards of thrust on curved surface
= weight of fluid contained,

so that

thrust on base is less than weight of fluid contained.

(3) Finally, if the shape of the vessel is as in Fig. 25, the thrust of the fluid on the curved surface of the container has an *upward* component and we then have, resolving vertically,

thrust on base — resolved part of thrust on curved surface
= weight of fluid contained,

so that

thrust on base is greater than weight of fluid contained.

In fact, in these three cases, if the areas of the bases are the same and the liquid is at the same height, the total thrust on the base in each case is the same.

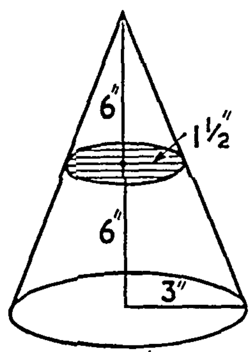


Fig. 26.

Example 1.—*A hollow right circular cone of height* 12 *in. and radius of base* 3 *in. has its vertex uppermost and its base horizontal and contains sea-water weighing* 64 *lb. per cub. ft. to a height of* 6 *in. Find the total thrust on the base and the weight of the liquid.*

Let Fig. 26 represent the cone.

Then since the volume of a cone of height h and radius of base r, is $\frac{1}{3}\pi r^2 h$ we have

$$\text{volume of sea-water} = \frac{1}{3}\pi\left(\frac{3}{12}\right)^2 . \frac{12}{12} - \frac{1}{3}\pi\left(\frac{1\frac{1}{2}}{12}\right)^2 . \frac{6}{12} \text{ cub. ft.}$$

$$= \frac{1}{3}\pi\left(\frac{1}{16} - \frac{1}{128}\right) \text{ cub. ft.}$$

$$= \frac{1}{3}\pi \, . \frac{7}{128} \text{ cub. ft.}$$

$$\therefore \quad \text{Weight of liquid} = \frac{1}{3}\pi \frac{7}{128} .64 \text{ lb. wt.}$$

$$= \frac{7\pi}{6} \text{ lb. wt.}$$

$$= 3{\cdot}665 \text{ lb. wt.}$$

But the pressure-intensity at the base

$$= \frac{64 \times 6}{12} \text{ lb. wt. per sq. ft.};$$

$$\text{total thrust on base} = \frac{64 \times 6}{12} \times \pi (\tfrac{1}{4})^2 \text{ lb wt}$$
$$= 2\pi \text{ lb wt}$$
$$= 6\ 283 \text{ lb wt}$$

This conforms with case (3) above and it is clear that the difference between these two values, namely 2 618 lb wt, measures the vertical component of the thrust exerted by the liquid on the curved surface (see also Example 3 of § 29)

We shall consider the thrusts on curved surfaces in more detail in Chapter VII

26 The total thrust of a heavy homogeneous liquid on a plane area is equal to the product of the area and the pressure intensity at its centre of gravity

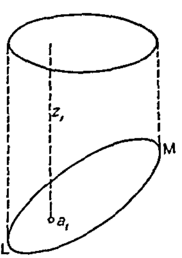


Fig 27

This theorem is very important as it enables the total thrust on a plane area to be determined if the depth of the centre of gravity (or centroid) of the area is known or if it can be calculated

Let LM represent any plane figure of area A sq ft immersed in a fluid (Fig 27) and suppose that a small element of it (of area a_1 sq ft) is at a depth z_1 ft below the surface

Then the pressure intensity at a_1 due to the weight of the liquid alone (i e neglecting atmospheric pressure) is ρz_1 lb wt per sq ft where ρ is the density of the liquid in lb per cub ft Thus we have

$$\text{thrust on element } a_1 = \rho z_1 a_1 \text{ lb wt approximately}$$

If we divide the remaining area into elements denoted by a_2 a_3 a_n at depths z_2 z_3 z_n then

$$\text{total thrust on figure} = \rho z_1 a_1 + \rho z_2 a_2 + \quad + \rho z_n a_n \text{ approximately}$$
$$= \rho \sum_{r=1}^{n} z_r a_r \text{ lb wt approximately,}$$

where the summation extends over the whole area since

$$a_1 + a_2 + \quad + a_n \text{ equals the whole area}$$

The reason this value of the total thrust is *only approximate* is that we have assumed that everywhere over the elemental area a_1

the pressure intensity is ρz_1 lb. wt. per sq. ft. In fact, different portions of the elemental area are at different depths and consequently the total thrust on the figure *exactly*

$$= \textit{limit} \text{ of } \rho z_1 a_1 + \rho z_2 a_2 + \ldots + \rho z_n a_n,$$

when the number of elements is indefinitely increased, but still subject to the condition

$$a_1 + a_2 + \ldots + a_n = \text{whole area},$$

i.e. we increase the number of elements and consequently decrease their individual areas.

Thus we may write exactly,

$$\text{total thrust on figure} = \lim_{n\to\infty} \rho \sum_{r=1}^{n} z_r a_r \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

But the expression $\Sigma z_r a_r$ was obtained in statics when finding the centre of gravity of a plane area (see *Tutorial Statics*, p. 221). Thus if $\bar{z}$ is the depth of the centre of gravity (or centroid) below the surface we have

$$A\bar{z} = \text{limit of } z_1 a_1 + z_2 a_2 + \ldots + z_n a_n \text{ as } n \to \infty$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} z_r a_r$$

and substituting in (i)

$$\text{total thrust on figure} = \rho A\bar{z} \text{ lb. wt.}$$

Now A is the area of the figure and $\rho\bar{z}$ is the pressure intensity at its centre of gravity, hence the theorem is proved.

27. Total thrust on simple geometrical figures immersed vertically in a fluid

We will now use the theorem of the last section to determine the total thrust on some simple geometrical shapes. In each case we shall only consider the thrust when the figure is immersed in the fluid vertically.

(1) *Rectangle with one edge in surface.*

Let sides be a, b.

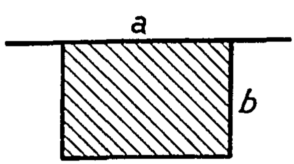


Fig. 28.

$$\text{Area of figure} = ab$$

$$\text{Depth of C G below surface} = \frac{b}{2},$$

$$\therefore \text{ Total thrust} = \rho\, ab\, \frac{b}{2}$$

$$= \tfrac{1}{2}\rho a b^2$$

(2) *Rectangle with upper edge horizontal at depth k*

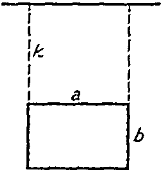


Fig 29

$$\text{Area of figure} = ab$$

$$\text{Depth of C G} = k + \frac{b}{2}$$

$$\therefore \text{ Total thrust} = \rho\, ab\left(k + \frac{b}{2}\right)$$

(3) *Triangle with one edge in surface*

Let ABC be triangle with $AC = a$ and let the altitude $BK = h$. If M is mid point of AC then G is centre of gravity of triangle where $GM = \frac{1}{3}BM$ (see *Tutorial Statics*, p 216)

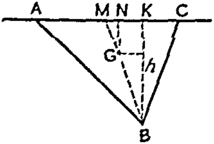


Fig 30

Hence, by similar triangles

$$GN = \tfrac{1}{3}BK$$
$$= \tfrac{1}{3}h,$$

and GN is the depth of C.G.

$$\text{Area of triangle} = \tfrac{1}{2}ah.$$
$$\text{Total thrust on triangle} = \rho.\tfrac{1}{2}ah.\tfrac{1}{3}h$$
$$= \tfrac{1}{6}\rho ah^2.$$

(4) *Triangle with upper edge horizontal and at depth k.*

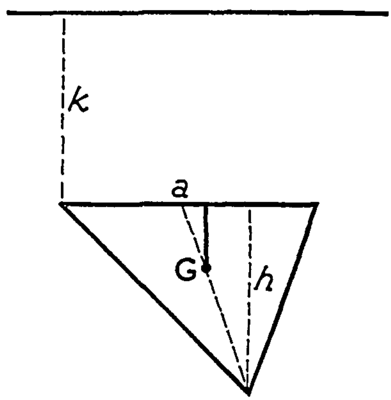


Fig. 31.

With notations as before,

$$\text{area of triangle} = \tfrac{1}{2}ah.$$
$$\text{Depth of C.G.} = k + \tfrac{1}{3}h;$$
$$\therefore \quad \text{Total thrust on triangle} = \rho.\tfrac{1}{2}ah\,(k + \tfrac{1}{3}h).$$

(5) *Triangle with lower edge horizontal and at depth k.*

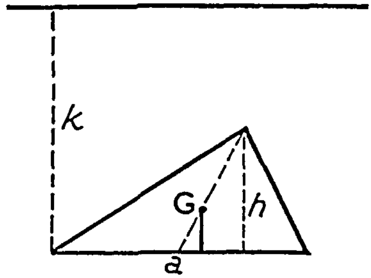


Fig. 32.

Area of triangle $= \frac{1}{2}ah$

Depth of C G $= k - \frac{1}{3}h$,

Total thrust on triangle $= \rho\ \frac{1}{2}ah\ (k - \frac{1}{3}h)$

(6) *Circle with one point of circumference in surface*

Let circle be of radius r

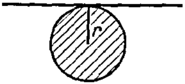


Fig 33

Area of circle $= \pi r^2$

Depth of C G $= r$

Total thrust on circle $= \rho\ \pi r^2\ r$

(7) *Circle with centre at depth k*

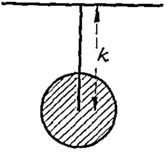


Fig 34

Area of circle $= \pi r^2$

Depth of C G $= k$,

Total thrust on circle $= \rho\ \pi r^2\ k$

(8) *Semicircle with bounding diameter in surface*

Let radius be r

Fig 35

$$\text{Area of semicircle} = \frac{\pi r^2}{2}.$$

$$\text{Depth of C.G.} = \frac{4r}{3\pi}\text{(see } \textit{Tutorial Statics}\text{, p. 318).}$$

$$\therefore \quad \text{Total thrust on semicircle} = \frac{\rho . \pi r^2}{2} \cdot \frac{4r}{3\pi}$$

$$= \tfrac{2}{3}\rho r^3.$$

(9) *Semicircle with bounding diameter uppermost and horizontal at depth k.*

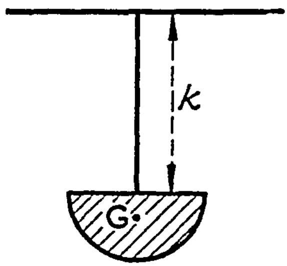


Fig. 36.

$$\text{Area of semicircle} = \frac{\pi r^2}{2}.$$

$$\text{Depth of C.G.} = k + \frac{4r}{3\pi};$$

$$\therefore \quad \text{Total thrust on semicircle} = \rho \cdot \frac{\pi r^2}{2}\left(k + \frac{4r}{3\pi}\right).$$

(10) *Semicircle with bounding diameter horizontal at depth k, curved edge being uppermost.*

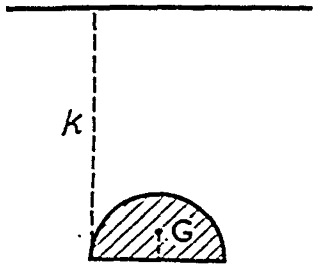


Fig. 37.

$$\text{Thrust on triangle ABD} = \rho\ \tfrac{1}{2}ab\ \frac{b}{3}$$
$$= \tfrac{1}{6}\rho ab^2$$
$$\text{Thrust on triangle BCD} = \rho\ \tfrac{1}{2}ab\ \tfrac{2}{3}b$$
$$= \tfrac{1}{3}\rho ab^2,$$

since the C G of triangle BCD is $\frac{2}{3}b$ below the surface

Hence $$\frac{\text{thrust on } \triangle \text{ BCD}}{\text{thrust on } \triangle \text{ ABD}} = \frac{\frac{1}{3}\rho ab^2}{\frac{1}{6}\rho ab^2} = 2$$

so that $$\text{thrust on } \triangle \text{ BCD} = 2 \times \text{thrust on } \triangle \text{ ABD}$$

Example 3 —*A rectangular area* ABCD *has the side* AB *in the surface of water, the side* AD (10 *feet long*) *being vertical and submerged, divide the area by horizontal lines into three parts, on each of which the thrust is the same*

Let Fig 41 represent the rectangular area

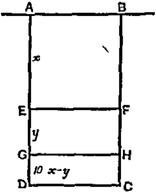


Fig 41

Let EF and GH be the lines of division Let AE $= x$ ft, EG $= y$ ft

Then $\qquad$ CD $= (10 - x - y)$ ft

Thrust on ABFE $=$ *area of ABFE* $\times$ *pressure intensity at C G*

of area $$= x \text{ AB} \times w\frac{x}{2}$$

Similarly, thrust on EFHG $= y$ AB $\times\, w\left(x + \frac{y}{2}\right)$,

and thrust on GHCD $= (10 - x - y)\,\text{AB} \times w\left(x + y + \frac{10 - x - y}{2}\right)$

Equating these we get

$$\frac{x^2}{2} = y\left(x + \frac{y}{2}\right) = (10 - x - y)\left(x + y + \frac{10 - x - y}{2}\right),$$

$$\therefore \quad x^2 = y(2x + y) = (10 - x - y)(10 + x + y) = 100 - (x + y)^2.$$

From the first two we have $\qquad x^2 - 2xy - y^2 = 0.$

From the first and last we have $\; 2x^2 + 2xy + y^2 = 100.$

Adding we get $3x^2 = 100$ or $x = \frac{10}{3}\sqrt{3}$.

Also $(x + y)^2 = 100 - x^2 = 100 - \frac{100}{3} = \frac{200}{3}$;

$$\therefore \quad x + y = \frac{10\sqrt{2}}{\sqrt{3}} = \frac{10}{3}\sqrt{6};$$

$$\therefore \quad y = \tfrac{10}{3}(\sqrt{6} - \sqrt{3}).$$

Thus $x = 5{\cdot}77$ ft., $y = 2{\cdot}39$ ft., and $10 - x - y = 1{\cdot}84$ ft.

Example 4.—*A semicircular lamina with centre* O *and diameter* AOB *is immersed vertically in a liquid with its plane vertical and the*

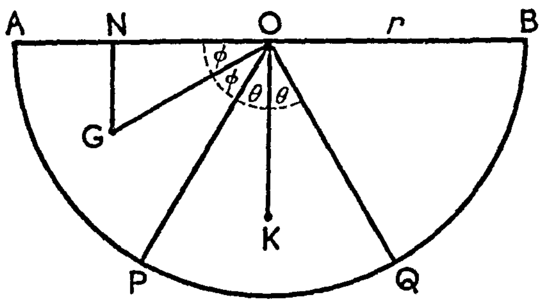


Fig. 42.

diameter AOB *in the free surface.* OP *and* OQ *are radii such that the angles* AOP, QOB *are each* 2ϕ *and* POQ *is* 2θ. *If the liquid thrust on each of the sectors* AOP, POQ, QOB *is the same, show that* $\sin\theta = \sin^2\phi$.

Deduce, or prove directly, that $\sin\theta = \frac{1}{3}$. (*Inter. Sc.*)

Let Fig. 42 represent the lamina and suppose that G, K are the centres of gravity (or centroids) of two adjacent sectors as shown. Let GN be drawn perpendicular to AB. Then since AB is horizontal (in surface of liquid), GN is vertical.

If r is the radius of the semicircle,

$$\text{OG} = \tfrac{2}{3}r\,\frac{\sin\phi}{\phi},$$

and

$$\text{OK} = \tfrac{2}{3}r\,\frac{\sin\theta}{\theta}.$$

OK is perpendicular to AB and is therefore vertical.

We may now write down the thrusts on the three sectors If ρ is the density of the fluid we have —

$$\begin{aligned}\text{thrust on sector BOQ} &= \text{thrust on sector AOP (by symmetry)}\\ &= \rho \times \text{area sector AOP} \times \text{GN}\\ &= \rho\ \tfrac{1}{2}r^2(2\phi)\ \text{OG}\sin\phi \text{ since GN} = \text{OG}\sin\phi,\\ &= \rho\ \tfrac{1}{2}r^2(2\phi)\ \tfrac{2}{3}r\frac{\sin\phi}{\phi}\ \sin\phi,\end{aligned}$$

using the value of OG above,

$$= \tfrac{2}{3}\rho r^3\sin^2\phi \qquad \text{(i)}$$

Also, thrust on sector POQ $= \rho \times$ area sector POQ $\times$ OK

$$= \rho\ \tfrac{1}{2}r^2(2\theta)\ \tfrac{2}{3}r\frac{\sin\theta}{\theta},$$

using the value of OK above,

$$= \tfrac{2}{3}\rho r^3\sin\theta \qquad \text{(ii)}$$

But the liquid thrusts on all the sectors are the same, hence equating (i) and (ii),

$$\sin\theta = \sin^2\phi$$

We may prove directly that $\sin\theta = \frac{1}{3}$ as follows —

Since the thrust on each sector is the same, three times the thrust on either must equal the total thrust on the semicircle which we may find independently A semicircle is a sector of angle 2α where $\alpha = \pi/2$, so that from the given formula, the C G of a semicircle is at a distance

$$\tfrac{2}{3}r\frac{\sin\pi/2}{\pi/2} \text{ from the centre,}$$

i e

$$\frac{4r}{3\pi} \qquad \text{from the centre}$$

Thus the thrust on the whole semicircle $= \rho\ \frac{1}{2}\pi r^2\ \frac{4r}{3\pi}$ (iii)

and this must equal three times the thrust on either sector,

putting (iii) equal to three times (ii),

$$\rho\ \tfrac{1}{2}\pi r^2\ \frac{4r}{3\pi} = 3\ \tfrac{2}{3}\rho r^3\sin\theta,$$

giving $\sin\theta = \frac{1}{3}$

Incidentally, we may *deduce* this result since we know that

$$\sin\theta = \sin^2\phi \text{ and also that } \theta + 2\phi = 90^\circ,$$

$$\phi = 45 - \tfrac{1}{2}\theta,$$

$$\sin\phi = \frac{1}{\sqrt{2}}\cos\tfrac{1}{2}\theta - \frac{1}{\sqrt{2}}\sin\tfrac{1}{2}\theta,$$

$$\therefore \quad \sin^2 \phi = \tfrac{1}{2}(\cos^2 \tfrac{1}{2}\theta - 2\sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\theta)$$
$$= \tfrac{1}{2}\left(1 - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$
$$= \tfrac{1}{2}(1 - \sin\theta);$$
$$\therefore \quad \sin\theta = \sin^2\phi = \tfrac{1}{2}(1 - \sin\theta),$$

so that $\quad 2\sin\theta = 1 - \sin\theta,$

or $\quad 3\sin\theta = 1,$

giving $\quad \sin\theta = \tfrac{1}{3}.$

28. Total thrust on a combination of simple geometrical figures immersed vertically in a fluid

We may extend § 27 to find the total thrust on an area which can

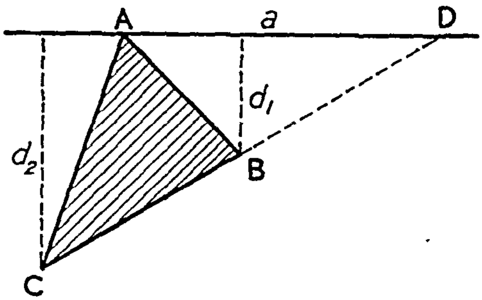


Fig. 43.

be considered as the sum or difference of two or more simple shapes. This is best illustrated by actual examples.

Example 1.—*A triangle* ABC, *immersed vertically in a fluid of density* ρ, *has its vertex* A *in the surface and* B, C *are at depths* d_1, d_2 *respectively. If* CB *produced meets the surface in* D, *find the total thrust of the fluid pressure on the triangle* ABC *in terms of* d_1, d_2, a, *where* $a =$ AD.

In this case (see Fig. 43) we have no need to find the C.G. of the triangle ABC because we can find directly the total thrusts on the triangles ACD and ABD and subtract.

Thus, $\quad$ area of $\triangle$ ABD $= \tfrac{1}{2}ad_1$;

$$\therefore \quad \text{thrust on } \triangle \text{ ABD} = \rho \,.\, \tfrac{1}{2}ad_1 \,.\, \frac{d_1}{3}$$
$$= \tfrac{1}{6}a\rho d_1^2.$$

Again, $\quad$ area of $\triangle$ ACD $= \tfrac{1}{2}ad_2$;

$$\therefore \quad \text{thrust on } \triangle \text{ ACD} = \rho \,.\, \tfrac{1}{2}ad_2 \,.\, \tfrac{1}{3}d_2$$
$$= \tfrac{1}{6}a\rho d_2^2.$$

Consequently,

$$\begin{aligned}\text{thrust on } \triangle\, ABC &= \text{thrust on } \triangle\, ACD - \text{thrust on } \triangle\, ABD \\ &= \tfrac{1}{6}a\rho d_2^2 - \tfrac{1}{6}a\rho d_1^2 \\ &= \tfrac{1}{6}a\rho\,(d_2^2 - d_1^2)\end{aligned}$$

Example 2 —*Find the total thrust on a trapezium having its parallel sides (of lengths a, b and distance d apart) horizontal, the shorter (a) being uppermost and at a depth k.*

Let ABCD (Fig 44) represent the trapezium then if we draw perpendiculars AN, BM we split up the trapezium into a rectangle and two triangles

Let $DN = x$, then $MC = b - a - x$ To find the depth ($\bar{z}$) of the centre of gravity of the trapezium we require

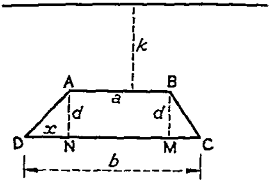


Fig 44

$$\begin{aligned}\text{area of } \triangle\, AND &= \tfrac{1}{2}xd, \\ \text{depth of C G of } \triangle\, AND &= k + \tfrac{2}{3}d, \\ \text{area of rectangle } ABMN &= ad, \\ \text{depth of C G of this rectangle} &= k + \frac{d}{2} \\ \text{area of } \triangle\, BMC &= \tfrac{1}{2}d\,(b - a - x) \\ \text{depth of C G of } \triangle\, BMC &= k + \tfrac{2}{3}d\end{aligned}$$

The equation for finding the C G of the trapezium is

area trap $\times\, \bar{z}$ = area $\triangle$ AND $\times$ its depth of C G + area rect $\times$ its depth of C G + area $\triangle$ BMC $\times$ its depth of C G

Since the area of the trapezium is $\frac{1}{2}(a+b)\,d$, this becomes

$$\begin{aligned}\tfrac{1}{2}(a+b)\,d.\bar{z} &= \tfrac{1}{2}xd\,(k + \tfrac{2}{3}d) + ad\,(k + \tfrac{1}{2}d) + \tfrac{1}{2}d\,(b - a - x)(k + \tfrac{2}{3}d) \\ &= \tfrac{1}{2}d\,(k + \tfrac{2}{3}d)(x + b - a - x) + ad\,(k + \tfrac{1}{2}d) \\ &= \tfrac{1}{2}d\,(bk + \tfrac{2}{3}bd - ak - \tfrac{2}{3}ad) + ad\,(k + \tfrac{1}{2}d)\end{aligned}$$

$$= \tfrac{1}{2}bdk + \tfrac{1}{3}bd^2 - \tfrac{1}{2}adk - \tfrac{1}{3}ad^2 + adk + \tfrac{1}{2}ad^2$$
$$= \tfrac{1}{2}adk + \tfrac{1}{2}bdk + \tfrac{1}{6}ad^2 + \tfrac{1}{3}bd^2$$
$$= \frac{d}{6}(3ak + 3bk + ad + 2bd).$$

Notice that we have no need to simplify the left hand side, because, since we require the total thrust on the area which is $\rho \times$ area $\times \bar{z}$, it is better to leave the left hand side alone since it is, in fact, the area $\times \bar{z}$.

Thus, total thrust $= \rho \times$ right hand side

$$= \frac{\rho d}{6}\Big[3k(a+b) + d(a+2b)\Big].$$

Example 3.—*A triangular area* ABC *is immersed in liquid with the vertex* A *in the surface and* B, C *at depths* h, k *respectively. If a horizontal line through* B *divides the triangle into two parts on which the resultant thrusts of the liquid are equal, prove that*

$$h/k = (1+\sqrt{17})/8,$$

atmospheric pressure being neglected. (*Inter. Sc.*)

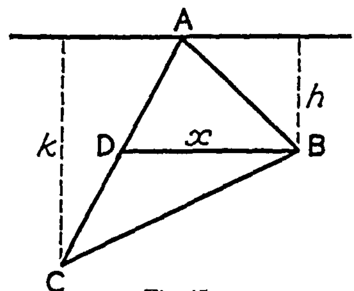


Fig. 45.

Let Fig. 45 represent the triangle and suppose the horizontal line through B cuts the edge AC in D. Let BD $= x$. In this case we have to find the thrust on each of the triangles ABD, CBD and equate.

$$\text{Area of } \triangle \text{ ABD} = \tfrac{1}{2}hx.$$
$$\text{Depth of C.G. of } \triangle \text{ ABD} = \tfrac{2}{3}h;$$
$$\therefore \quad \text{Thrust on } \triangle \text{ ABD} = \rho . \tfrac{1}{2}hx . \tfrac{2}{3}h$$
$$= \tfrac{1}{3}\rho x h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$
$$\text{Area of } \triangle \text{ BDC} = \tfrac{1}{2}x(k-h).$$
$$\text{Depth of C.G. of } \triangle \text{ BDC} = h + \frac{k-h}{3}$$
$$= \frac{2h+k}{3};$$
$$\therefore \quad \text{Thrust on } \triangle \text{ BDC} = \rho . \tfrac{1}{2}x(k-h)\frac{(2h+k)}{3}$$
$$= \tfrac{1}{6}\rho x(k-h)(2h+k) \quad \ldots\ldots \text{(ii)}$$

But the thrusts on the two triangles are equal therefore (i) = (ii),

$$\tfrac{1}{3}\rho x h^2 = \tfrac{1}{6}\rho x (k-h)(2h+k)$$

$$h^2 = \tfrac{1}{2}(2hk - 2h^2 + k^2 - hk)$$

$$2h^2 = hk + k^2 - 2h^2,$$

$$4h^2 - hk - k^2 = 0$$

Solving this by the rule for quadratic equations, we have

$$h = \frac{k \pm \sqrt{(k^2 + 4\cdot 2\cdot 2k^2)}}{8}$$

$$= k\,\frac{(1 \pm \sqrt{17})}{8},$$

$$\frac{h}{k} = \frac{1+\sqrt{17}}{8}$$

as the negative value is inadmissible

✓ **Example 4** —*A rectangle* ABCD *is immersed vertically in a liquid with* AB *in the water line If* E *is a point in* BC *such that the liquid thrusts on the areas* CDE *and* ABED *are equal, show that* BE EC $= 1 \quad \sqrt{3}$ (*Inter Sc*)

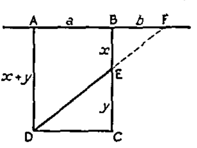


Fig 46

Let BE $= x$ and EC $= y$ (Fig 46) then we have to find the total thrust on the trapezium ABED and on the triangle CDE

We may find the thrust on the trapezium either by considering it as a rectangle + a triangle or by producing DE to cut the surface in F and taking the difference in total thrust on the triangles ADF, BEF If we put AB $= a$ BF $= b$ then

thrust on trap ABED = thrust on △ ADF — thrust on △ BEF

$$= \rho\,\tfrac{1}{2}(a+b)(x+y) \times \frac{(x+y)}{3} - \rho\,\frac{1}{2}bx \times \frac{x}{3} \quad \text{(i)}$$

and since triangles BEF, DEC are similar, $\dfrac{b}{x} = \dfrac{a}{y}$, $\quad b = \dfrac{ax}{y}$

Substituting for b in (i) we have

$$\text{thrust on trap ABED} = \frac{1}{6}\rho\left(a + \frac{ax}{y}\right)(x+y)^2 - \frac{1}{6}\rho x^2\,\frac{ax}{y}$$

$$= \frac{a\rho}{6y}\left[(x+y)^3 - x^3\right]$$

$$= \frac{a\rho}{6}(3x^2 + 3xy + y^2) \quad \text{(ii)}$$

Also, area of $\triangle$ CDE $= \frac{1}{2}ay$,

depth of C.G. of $\triangle$ CDE $= x + \frac{2}{3}y$;

$\therefore$ thrust on $\triangle$ CDE $= \rho . \frac{1}{2}ay\,(x + \frac{2}{3}y)$

$$= \frac{a\rho}{6}(3xy + 2y^2) \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Equating (ii) and (iii), since the thrusts are given equal,

$$\frac{a\rho}{6}(3x^2 + 3xy + y^2) = \frac{a\rho}{6}(3xy + 2y^2);$$

$$\therefore \quad 3x^2 + y^2 = 2y^2;$$

$$\therefore \quad 3x^2 = y^2;$$

$$\therefore \quad \frac{x}{y} = \frac{1}{\sqrt{3}};$$

$$\therefore \quad \text{BE} : \text{EC} = 1 : \sqrt{3}.$$

29. Total thrusts on plane, non-vertical areas in contact with a fluid

The theorem of § 26 was proved to be true, independently of whether the plane area was vertical or not. The total thrust depends only on the area of the figure and the pressure intensity at its centre of gravity. Thus exactly the same procedure may be adopted as in § 26 to find the total thrust.

Example 1.—*To find the total thrust of water on the slanting face of an embankment* 100 *metres long and* 30 *metres broad, which shelves down to a depth of* 12 *metres below the surface at the lowest part.*

The area exposed to pressure $= 100 \times 30 = 3000$ sq. m.

The depth of its C.G. $= \frac{1}{2}$ depth of lowest portion

$= 6$m. $= 600$ cm.;

$\therefore$ pressure intensity at C.G. $= 600$ grm. wt. per sq. cm.

$= 600 \times 100 \times 100$ grm. wt. per sq. m.

$= 6000$ kilog. wt. per sq. m.;

$\therefore$ total thrust $= 6000 \times 3000$ kilog. wt.

$= 18{,}000{,}000$ kilog. wt.

Example 2.—*A rectangular area is immersed in fluid with two edges horizontal such that its plane makes an angle θ with the vertical. To find the total thrust.*

Let the rectangle have sides of length a, b Then if the upper edge (a) is at a depth k below the surface then the C G of the area is at a depth of $k + \frac{b}{2}\cos\theta$ Fig 47 shows a section of the figure

Therefore, the total thrust $= \rho\, ab\left(k + \frac{b}{2}\cos\theta\right)$

Example 3 —*A hollow right pyramid has a square base of side 1 ft and is 1 ft in height Find the total thrust on the base and each of the four faces, if the pyramid be filled with water weighing 62 4 lb wt per cub ft*

Since the area of the base is 1 sq ft and the depth of the C G of the base is 1 ft from the vertex, we have total thrust (T) on base $= 62\ 4$ lb wt

Let ABC (Fig 48) represent a section of the pyramid made by a *vertical plane cutting the base in a line parallel to two of its edges*

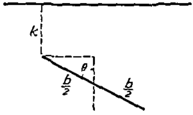


Fig 47

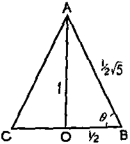


Fig 48

Then if O is the centre of the square base AO = 1 ft, OB = $\frac{1}{2}$ ft and therefore AB = $\frac{1}{2}\sqrt{5}$ ft

If θ is $\angle$ ABO $\cos\theta = \frac{1}{\sqrt{5}}$

Now AB is a line of greatest slope in a triangular face,

area of each triangular face $= \frac{1}{2} \times \frac{1}{2}\sqrt{5} = \frac{1}{4}\sqrt{5}$ sq ft

and the depth of C G of each triangular face $= \frac{2}{3} \times 1$ ft,

total thrust (F) on each triangular face $= \frac{2}{3} \times \frac{1}{4}\sqrt{5} \times 62\ 4$ lb wt

$= (10\ 4)\sqrt{5}$ lb wt

Now the volume of the pyramid is $\frac{1}{3}$ 1 1 cub ft, therefore the weight of water (W) contained in the pyramid is $\frac{1}{3}$ (62 4) = 20 8 lb wt

It is very important to realise exactly how it comes about that the weight of water is 20 8 lb wt, yet the total thrust on the base is 62 4 lb wt We may examine this more closely by means of a diagram and verify by resolving all the forces vertically Before any

Resolving these forces vertically, for the equilibrium of the container we have

$$P + 4F \cos\theta = T,$$

which is the same as (i) above, since P (the tension in the rope) is clearly equal to the weight of fluid (W)

A very important corollary of the theorem concerning the total thrust on an immersed plane area is this that the total thrust on the area depends only on the area and the depth of its C G , and is therefore unaltered by turning the area about its C G provided that the whole of the area is kept below the surface

30 Total thrust on plane areas partly in several fluids which do not mix

When part of an area is in contact with one fluid and another part in contact with a fluid of different density then we must find the total thrust on each part and add

Example 1 —*A liquid of density ρ_1 has a second liquid (density ρ_2) superimposed on it to a depth of d ft If a rectangle of sides a b ($b > d$) is immersed vertically with a side of length a in the surface of the upper liquid, find the total thrust on the rectangle*

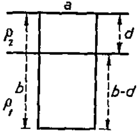


Fig 51

Let Fig 51 represent the rectangle The common surface of the two liquids divides the rectangle into two smaller rectangles, the upper of area ad and the lower of area $a(b-d)$

The C G of the upper rectangle is at a depth $\frac{d}{2}$ therefore the *pressure intensity* at the C G of the upper rectangle is $\rho_2 \frac{d}{2}$,

$$\text{thrust on upper rectangle} = \rho_2 ad \frac{d}{2}$$

The C G of the lower rectangle is at a depth $d + \frac{1}{2}(b-d)$ below the free surface and the *pressure intensity* at the C G of the lower rectangle is

$$\rho_2 d + \rho_1 \frac{(b-d)}{2}$$

$$\text{thrust on lower rectangle is } a(b-d)\left[\rho_2 d + \rho_1 \frac{(b-d)}{2}\right],$$

total thrust on rectangle is

$$\tfrac{1}{2}a\rho_2 d^2 + a\rho_2 d(b-d) + \tfrac{1}{2}a\rho_1 (b-d)^2$$
$$= \tfrac{1}{2}a\left[\rho_1 (b-d)^2 + \rho_2 d(2b-d)\right]$$

31. Total thrust on a plane area found independently of the position of the C.G.

The theorem of § 26 gave the total thrust on a plane figure in terms of the pressure intensity at the centre of gravity and the area of the figure. Thus if the position of the C.G. of the area is known or can easily be calculated, and the total area is known, then the total thrust on the area can be found immediately. It must be borne in mind, however, that we may find the total thrust directly, by finding the thrust on a small portion or element of the area and then finding the limit of the sum of the thrusts on all such elements.

We will now work out an example two ways, finding the total thrust directly, (*a*) using the calculus and (*b*) without the calculus.

Example 1.—*To determine directly the total thrust on a rectangle immersed vertically in a liquid (density ρ) with the upper edge horizontal at a depth k* [cf. § 27 (2)].

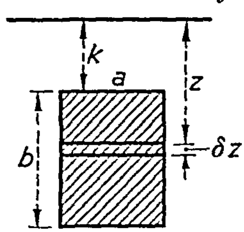


Fig. 52.

(*a*) *Using calculus.*

Let the rectangle have sides a, b. Consider a horizontal strip: width δz, depth z (Fig. 52).

Then area of strip $= a.\delta z$;

$\therefore$ thrust on strip $= \rho.a\delta z.z$;

$\therefore$ total thrust on rectangle $= \int \rho azdz$,

where the integration covers all values of z to include the whole rectangle, *i.e.* from $z = k$ to $z = k + b$.

Thus, total thrust on rectangle $= \int_k^{k+b} \rho azdz$

$$= \left[\frac{\rho az^2}{2}\right]_k^{k+b}$$

$$= \frac{\rho a}{2}\left[(k+b)^2 - k^2\right]$$

$$= \frac{\rho a}{2}(2kb + b^2)$$

$$= \rho ab\left(k + \frac{b}{2}\right).$$

(*b*) *Without calculus.*

Let sides of rectangle be a, b as before. Divide the rectangle into n horizontal strips each of width l so that

$$nl = b.$$

We shall later increase the number of strips and decrease their width so that nl will still be equal to b.

Let
$$CD = b_r$$
From the similar triangles OCD, OLM,
$$\frac{CD}{z_r} = \frac{LM}{h}$$
i e
$$\frac{b_r}{z_r} = \frac{LM}{h}$$
$$= \frac{a_r}{h} \text{ since } a_r = AB = LM,$$
$$a_r z_r = hb_r,$$

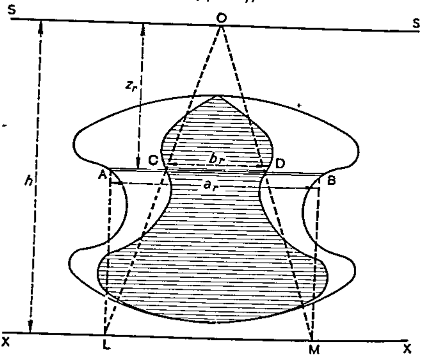


Fig 54

(1) becomes
$$\text{total thrust on area} = \lim_{n \to \infty} \Sigma_{r=1}^{n} \rho l h b_r$$
$$= \rho h \lim_{n \to \infty} \Sigma_{r=1}^{n} b_r l$$
But $\lim_{n \to \infty} \Sigma_{r=1}^{n} b_r l$ is the shaded area (A_1),
$$\text{total thrust on area} = \rho h A_1$$
Thus by finding the area A by counting squares and measuring h, we obtain the total thrust on the area The shaded area, obtained

as we have indicated, is called the *first derived figure* for the original area. If very high accuracy is necessary the area of the first derived figure is found by means of an instrument called the *planimeter*.

This method is valid whatever the shape of the original figure, symmetrical or not, and wherever the point O is chosen on the line SS, since the triangles OCD, OLM will always be similar.

It is worth while noticing that the graphical method we have adopted here is also convenient for finding the position of the centre of gravity of an irregular figure, for if its area is A_0 and its centre of gravity a distance $\bar{z}$ below the surface, we have by § 26

$$\text{total thrust on figure} = \rho A_0 \bar{z}$$
$$= \rho A_1 h, \text{ as above};$$
$$\therefore \quad A_0 \bar{z} = A_1 h;$$
$$\therefore \quad \bar{z} = \frac{A_1 h}{A_0}$$

or $\bar{z} = \dfrac{\text{area of first derived figure}}{\text{area of original figure}} \times h.$

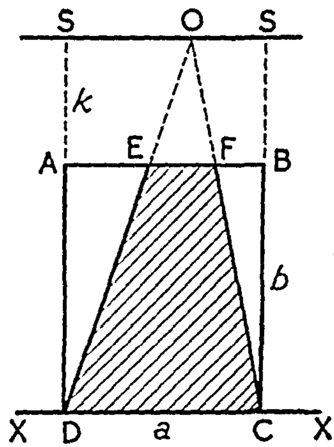


Fig. 55.

Example 1.—*Find the total thrust on a rectangle immersed vertically with its upper edge horizontal and at a depth k.*

If the original figure immersed has edges horizontal or vertical the first derived figure will be a straight-sided polygon and we may use this fact to determine the total thrust on the rectangle. Of course the value of this graphical method is that it applies to areas for which the total thrust cannot be found conveniently by any other method, but this example will be useful for confirmation of the method.

Let ABCD (Fig. 55) be the rectangle of sides a, b as shown. Let O (the pole) be any point in the surface line SS. Then the shaded figure EFCD is the first derived figure. Its area is $\frac{1}{2}(\text{EF} + \text{CD})\, b$, since it is a trapezium, and from the similar triangles OEF, ODC we have

$$\frac{\text{EF}}{k} = \frac{\text{CD}}{b + k};$$
$$\therefore \quad \text{EF} = \frac{ak}{b + k};$$
$$\therefore \quad \text{area of first derived figure} = \tfrac{1}{2} b \left(a + \frac{ak}{b + k}\right)$$
$$= \tfrac{1}{2} ab \left(\frac{b + 2k}{b + k}\right);$$

$$\text{total thrust on rectangle ABCD} = \rho (b + k)\ \tfrac{1}{2}ab \left(\frac{b + 2k}{b + k}\right)$$
$$= \rho ab \left(k + \frac{b}{2}\right)$$

which agrees with the result of § 27 (2)

Exercises IV

(1) Determine the thrust in pounds weight on every foot breadth of a vertical wall of a rectangular reservoir of water 150 ft deep

(2) A lock gate, 10 ft wide and 10 ft deep, has water on one side 8 ft deep and on the other 5 ft deep, in each case measured from the lower edge of the gate Determine the resultant thrust

(3) Determine the total thrust on one side of a rectangular vertical dock gate 50 ft wide immersed in sea water to a depth of 25 ft, having given that the specific gravity of sea water is 1 026

(4) A cube whose edge is 1 ft is suspended in water with its upper face horizontal, at a depth of $2\frac{1}{2}$ ft below the surface Find the liquid thrust on each face of the cube

(5) An artificial lake, $\frac{1}{4}$ ml long and 100 yd broad, with a gradually shelving bottom varying from nothing at one end to 88 ft at the other, is dammed at the deep end by a masonry wall across its entire breadth Find the total thrust on the embankment when the lake is full of water weighing $\frac{3}{4}$ ton to the cubic yard Find also the total weight of water in the lake

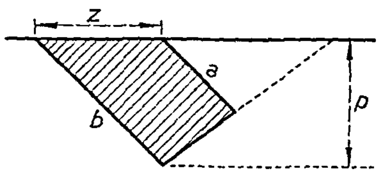
(6) A hollow cone stands with its base on a horizontal table The area of the base is a sq in, and the height h in, its weight is equal to the weight of water it will contain When filled with water, what is the ratio of the thrust of the water on the base to that of the base on the table? (The volume of a cone is one third of that of a cylinder with the same base and altitude)

(7) A right circular cone is open at the base and has a small hole at the vertex, it rests on a horizontal plane, the diameter of the base being 1 ft and the height of the cone 2 ft Find the weight of the cone that it may be just possible to fill it with water without causing it to lift from the plane

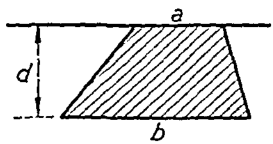
(8) The surface of a vessel containing liquid consists of a number of plane faces of areas a_1 a_2, a_3 , etc, and the centres of gravity of these areas are at depths z_1, z_2, z_3 , etc, below the surface of the liquid Write down the sum of the resultant thrusts on the several faces the weight of a unit volume of liquid being w, and show that this sum is equal to the product of the whole superficial area of the vessel into the pressure intensity at the centre of gravity of this area

(9) One end of a water conduit is closed by a watertight square plate ABCD, whose plane is vertical and with AB in the surface of the water. Show how to divide the plate into two parts by a line through A so that the thrusts on the two parts may be equal. (Inter. Eng.)

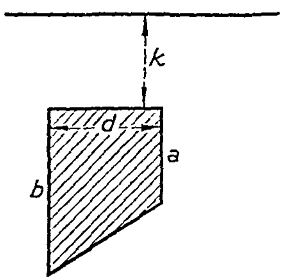
(10) In each of the four cases below, the lamina is immersed vertically in a liquid of density ρ. Find the total liquid thrust on one side of the lamina in terms of the given quantities.



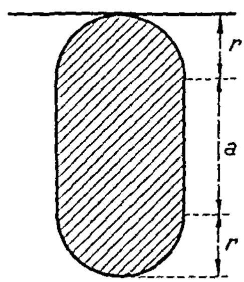
(i)



(ii)



(iii)



(iv)

In each case, deduce the total liquid thrust when $a = 0$, and confirm that the results agree with those already obtained in § 27.

(11) A swimming bath measures 100 ft. by 30 ft. At the shallow end the depth is 3 ft., while at the deep end it is 12 ft., and between the two the depth increases uniformly. Find the total thrust, in tons weight, on the base and on each side, given that 4 cub. yd. of water weigh 3 ton.

(12) A rectangle ABCD is immersed in a liquid so that AD is on the surface. AB is divided at P and Q so that the thrust exerted by the liquid on the triangles ADP, PDQ, QDB are all equal. Find the ratios AP : PQ : QB. (High. Sch. I.)

(13) A swimming bath is 90 ft. long. At the deep end it is 9 ft. deep and at the shallow end 2 ft. 6 in. The bottom slopes steadily downward from the shallow end for the first 45 ft. of the length, then remains horizontal at a depth of 6 ft. until 20 ft. from the deep end when it drops vertically to the depth of 9 ft. forming the diving pool. Find the thrust, in tons weight, on one side of the bath, assuming 1 cub. ft. of water weighs 62·5 lb. (High. Sch. I.)

(14) ABC is a triangular lamina immersed vertically in liquid with AB in the surface P and Q are points in BC such that the thrusts due to the liquid on the triangles ABP, APQ and AQC are equal Show that

$$BP \quad BQ \quad BC = 1 \quad \sqrt{2} \quad \sqrt{3}$$

If PE be drawn parallel to CA to meet BA in E, find the ratio of the liquid thrust on the triangle EBP to that on the triangle ABC

(Inter Eng)

(15) A cube, of which the edge measures 1 ft, is immersed in water with one edge in the surface and each of the faces which meet in that edge inclined at 45° to the vertical Find the magnitude of the thrust of the water on each face of the cube, and the resultant thrust on the whole cube

Take the density of water to be 62 5 lb per cub ft

(High Sch I)

(16) A cube is filled with a liquid of weight W, and held with a diagonal vertical, find the total liquid thrust on one of the lower and one of the upper faces

(17) Two circles are drawn on a vertical wall of a reservoir, they touch each other externally, and the centre of one is vertically below the centre of the other the water in the reservoir rises so that the upper circle is just immersed Find the ratio of the radii when the resultant pressures on the two circles are equal

(18) A closed cubical vessel contains liquid It is held so that a diagonal of two of the faces is vertical and the free surface of the liquid bisects the two upper edges of the vertical faces Prove that the liquid thrust on one of the vertical faces is $\frac{25\sqrt{2}}{84}$ W, where W is the weight of the contained liquid

(Inter Eng)

(19) A triangular lamina, having an area of 4 sq ft, has its vertices immersed at depths of one, two, and three feet respectively in water Find the total thrust on the area, the atmospheric pressure being taken at 14 7 lb wt per sq in

(20) A layer of liquid of density ρ lb per cub ft and depth a ft is superimposed on a layer of liquid of density 2ρ and depth greater than $2a$ A square lamina of side $2a$ is immersed vertically with its upper edge (i) in the free surface, and (ii) in the surface common to the two liquids Find the total liquid thrust on the lamina in each case

(21) A rectangle with two sides vertical is drawn on a vertical face of a vessel containing water and oil (sp gr 0 9) The rectangle is just covered, so that it is divided into two parts by the common surface of the oil and the water If the thrusts due to the liquids on these two parts are equal, find the ratio of the depth of the oil to the height of the rectangle

(22) An equilateral triangle of side 3 in is immersed vertically in a liquid with one side in the surface If now a second liquid (of density half the former) is superimposed, find the depth of this liquid when the total liquid thrust on the triangle is double what it was originally

(23) By integration confirm the results of cases (3), (4), (5), (6) and (8) of § 27

(24) A circular lamina of radius 3 in. is immersed vertically in a liquid of density ρ lb. per cub. in., its centre being at a depth of 4 in. Find graphically the total liquid thrust on the lamina and compare the result with case (7) of § 27.

(25) A hollow cone, whose axis is vertical and base downwards, is filled with equal volumes of two liquids, whose densities are in the ratio of 3 : 1; prove that the total liquid thrust on the base is $(3 - \sqrt[3]{4})$ times as great as when the vessel is filled with the lighter liquid.

(26) A triangle ABC is immersed vertically in water with its vertex A in the surface and its base BC horizontal at a depth h. Show how to divide the area by horizontal lines into n strips on which the thrusts shall be equal.

ANSWERS

1. 703,125. **2.** 5·441 ton wt. **3.** 447·3 ton wt., nearly.

4. 2500 oz. wt., 3000 oz. wt., 3500 oz. wt. **5.** $32{,}266\frac{2}{3}$ ton wt., 484,000 ton wt.

6. 3 : 2. **7.** 65·45 lb.

9. If line through A cuts base in E, ED = 0·686a, where a is side of square.

10. (i) $\rho p^2 z(a^2 + ab + b^2)/6b^2$; (ii) $\rho d^2(a + 2b)/6$;
(iii) $\rho d[a^2 + ab + b^2 + 3k(a + b)]/6$; (iv) $\rho r(2r + a)(\pi r + 2a)/2$.

11. Base 627·47; ends 3·75, 60; side 87·5.

12. $1 : \sqrt{2} - 1 : \sqrt{3} - \sqrt{2}$. **13.** 47·135 ton wt. **14.** $1 : 3\sqrt{3}$.

15. Upper faces: 22·097 lb. wt.; lower faces: 88·388 lb. wt.; vertical faces: 44·194 lb. wt. Resultant thrust = 62·5 lb. wt. vertically upwards.

16. $2W/\sqrt{3}$, $W/\sqrt{3}$. **17.** $\frac{1}{2}(1 + \sqrt{5}) : 1$ or 1·62 : 1.

19. 8967·2 lb. wt. **20.** (i) $5\rho a^3$ lb. wt.; (ii) $12\rho a^3$ lb. wt.

21. 0·71 : 1. **22.** $\sqrt{3}$ in. **24.** $113{\cdot}1\rho$ lb. wt.

26. The lines are at depths $h\left(\frac{1}{n}\right)^{\frac{1}{2}}, h\left(\frac{2}{n}\right)^{\frac{1}{2}}, \ldots, h\left(\frac{r}{n}\right)^{\frac{1}{2}}, \ldots$

CHAPTER V

PRESSURE OF HEAVY FLUIDS (3)

Centre of Pressure

33. Method of finding the position of the centre of pressure

We have already defined the centre of pressure of a plane area immersed in a fluid as the point where the resultant thrust meets the area. It must be clearly understood that although the position of the centre of gravity of the area could be used in finding the total thrust, yet the single force equal in magnitude to the total thrust, *i e* the resultant thrust, does *not* act at the centre of gravity but at the centre of pressure, which is always at a greater depth than the centre of gravity except, of course, when the area is horizontal (see § 42)

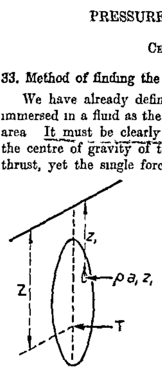


Fig 56

The general principle by which we may determine the depth of the centre of pressure in any particular case is as follows —

Let Fig 56 represent any plane area immersed vertically in a heavy homogeneous fluid of density ρ. Then, with the notation of § 26, if a_1 is the area of a small element at a depth z_1,

the thrust on element due to the fluid alone $= \rho a_1 z_1$, approximately, and consequently

total thrust (T) on area due to fluid alone

$$= \lim_{n \to \infty} \Sigma_{r=1}^{n} \rho a_r z_r \text{ exactly,}$$

$$= \rho \lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r, \text{ since } \rho \text{ is constant}$$

Now it is this limit which is equal to the magnitude of the resultant thrust (indicated by the large arrow) acting at the centre of pressure. Since this is the resultant of a number of parallel forces, to find its depth (*Z*) we take its moment about a line in the surface in the same vertical plane as the area and equate it to the sum of the moments of the constituent forces about the same line

Remember we are only considering the forces on one side of the area. Thus, moment about surface of thrust on element

$$= \rho a_1 z_1 . z_1, \text{ approximately},$$
$$= \rho a_1 z_1^2, \text{ approximately}.$$

Therefore the limit of the sum of the moments of all elemental thrusts

$$= \rho \lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

But the moment of the resultant thrust

$$= Z \times \rho \lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Equating (i) and (ii),

$$Z \times \rho \lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r = \rho \lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r^2$$

$$\therefore Z = \frac{\lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r^2}{\lim_{n \to \infty} \Sigma_{r=1}^{n} a_r z_r} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Thus the depth of the centre of pressure is independent of the density of the fluid (if homogeneous) since (iii) does not contain ρ.

Having determined the depth of the centre of pressure, we have therefore a horizontal line on which the centre of pressure must lie. We may fix its position on this line by taking a new axis and finding the limit of the sum of the moments of all elemental thrusts about this new axis and equating it to the total thrust multiplied by the distance of the centre of pressure from this new axis. This procedure may be avoided in cases where a single straight line can be drawn through the plane of the figure to bisect every *horizontal* strip. This line is then an axis of symmetry and the centre of pressure lies on it.

It will have been noticed that in this general case we supposed the area immersed *vertically* and we shall prove later on (§ 39) that there is no loss of generality in doing this.

Thus to determine the depth of the centre of pressure (Z) in any particular case we have to calculate the right hand side of (iii) for the area under consideration. The simplest method of doing this is to use calculus, but it may be found without (compare § 31), and in addition there is a geometrical method which is easily applicable in simple cases.

We propose here to find the position of the centre of pressure in two simple cases by each of the three methods indicated above.

34 Centres of Pressure using Calculus

Example 1 —*Find the position of the centre of pressure of a rectangle immersed vertically in a fluid with one side in the surface*

Let fluid be of density ρ If ABCD (Fig 57) represents the rectangle with sides a, b, consider an element of width δz at a depth z

$$\text{Area of element} = b\ \delta z$$

$$\text{Thrust on element} = \rho b\ \delta z\ z,$$

$$\text{moment about AB of thrust on element} = \rho b\ \delta z\ z \times z = \rho b z^2 \delta z$$

$$\text{total moment for all such elements} = \int_0^a \rho b z^2 dz$$

$$\text{and total thrust on rectangle} = \int_0^a \rho b z dz$$

Thus, if Z be the depth of the centre of pressure below the surface,

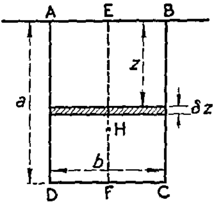


Fig 57

$$Z \times \int_0^a \rho b z dz = \int_0^a \rho b z^2 dz$$

$$Z = \frac{\int_0^a \rho b z^2 dz}{\int_0^a \rho b z dz} = \frac{\rho b \int_0^a z^2 dz}{\rho b \int_0^a z dz} = \left[\frac{1}{3}z^3\right]_0^a \Big/ \left[\frac{1}{2}z^2\right]_0^a = \tfrac{1}{3}a^3/\tfrac{1}{2}a^2 = \tfrac{2}{3}a$$

Therefore if E and F are the mid points of AB CD respectively, EF is a line of symmetry on which the centre of pressure lies so that the position of the centre of pressure (H) is given by

$$\text{EH} = \tfrac{2}{3}\text{EF} = \tfrac{2}{3}a$$

Example 2 —*Find the position of the centre of pressure of a triangle immersed vertically in a fluid with one side in the surface*

Let the side (AB) in the surface be of length a and let h be the depth of the vertex (C) below the surface (Fig 58)

Consider an element (DE) of length l and width δz at a depth z

$$\text{Area of element} = l\ \delta z$$

$$\text{Thrust on element} = \rho l\ \delta z\ z \qquad \text{(i)}$$

Moment about AB of thrust on element $= \rho l.\delta z.z^2$;

$\therefore$ total moment for all such elements $= \int_0^h \rho l z^2 dz$ .... (ii)

But the total thrust on the triangle $= \int_0^h \rho l z dz$ from (i);

$$\therefore \quad Z \times \int_0^h \rho l z dz = \int_0^h \rho l z^2 dz \quad \text{.............. (iii)}$$

A very important point to notice at this stage is that we cannot take l outside the integral sign because l is not constant for every elemental strip. In fact, l varies with z and consequently we must obtain an expression for l in terms of z before we can integrate. In the last example we could take b outside the integral signs because, there, b was constant for every element. Another point to notice in both

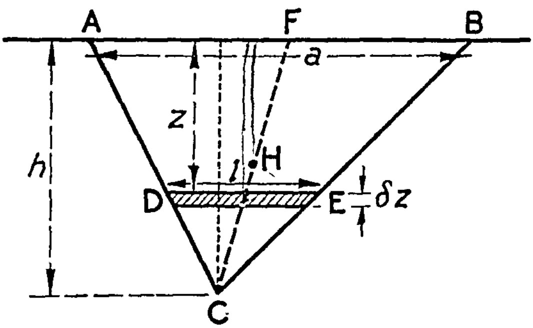


Fig. 58.

examples is that the limits for the integration are the values of z to be taken to include the whole area under consideration.

Thus, before proceeding further with (iii), we obtain l in terms of z and constants.

Triangles DEC, ABC are similar;

$$\therefore \quad \frac{DE}{h - z} = \frac{AB}{h}, \quad i.e. \quad \frac{l}{h - z} = \frac{a}{h};$$

$$\therefore \quad l = \frac{a}{h}(h - z).$$

Thus (iii) becomes

$$Z \times \int_0^h \frac{\rho a}{h}(h - z)\, z dz = \int_0^h \frac{\rho a}{h}(h - z)\, z^2 dz;$$

$$\therefore \quad Z = \frac{\frac{\rho a}{h}\int_0^h (hz^2 - z^3)\, dz}{\frac{\rho a}{h}\int_0^h (hz - z^2)\, dz},$$

is therefore $a_1 z_1^2$ Thus, if we require the centre of *gravity* of the solid wedge, its depth ($\bar{z}$) below the horizontal plane through AB (i e the surface of the liquid) is given by the limiting value of

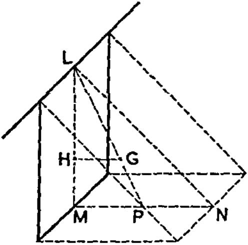


Fig 62

$$\bar{z}\ \Sigma a_r z_r = \Sigma a_r z_r^2 \qquad \text{(i)}$$

because lim $\Sigma a_r z_r$ is the total volume of the wedge, and lim $\Sigma a_r z_r^2$ is the total moment of the elemental volumes

Thus, from (i),

$$\bar{z} = \frac{\lim \Sigma a_r z_r^2}{\lim \Sigma a_r z_r} \qquad \text{(ii)}$$

Comparing this with the formula we obtained for the centre of pressure of a plane area, § 33 (iii), it is clear that the depth of the centre of *pressure* of the rectangle ABCD is the same as the depth of the centre of *gravity* of the solid wedge ABCDEF This position may be easily established

Let LMN be the central vertical triangular section of the solid wedge (Fig 62) If LP is a median of this triangle the centre of gravity of the solid wedge is at G, where LG = $\frac{2}{3}$LP, and as G projects horizontally into H, the centre of pressure of the rectangle LH = $\frac{2}{3}$LM, i e the depth of the centre of pressure of the rectangle is $\frac{2}{3}$ the length of the vertical edge

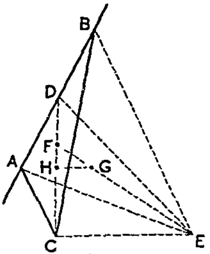


Fig 63

Example 2 —*Find the depth of the centre of pressure of a triangle immersed vertically in a fluid with one side in the surface*

The method is exactly the same as in the last example

Let ABC (Fig 63) represent the triangle of which CD is one median, AB being in the surface The solid in this case will be obtained by drawing CE horizontally and equal in length to the depth of C below the surface By joining AE, BE we have a solid tetrahedron whose

centre of gravity (G) projects into the centre of pressure (H) of the triangle ABC. If F is the centre of gravity of the triangle ABC, *i.e.* DE $= \frac{1}{3}$DC, then G is such that EG $= \frac{3}{4}$EF.

Therefore $\quad CH = \frac{3}{4}CF.$

But $\quad CF = \frac{2}{3}CD,$

$$CH = \tfrac{3}{4}.\tfrac{2}{3}CD;$$

$$\therefore \quad CH = \tfrac{1}{2}CD,$$

so that H is at a depth of half the depth below the surface of the vertex C.

We have devoted the last three sections to the problem of finding the position of the centre of pressure in two simple cases by three different methods. Of these three methods, that using calculus is by far the most important as it is of very wide application. The non-calculus method will become very laborious in more complicated cases, while the geometrical method cannot be applied universally since the positions of the centres of gravity of the resulting solids may neither be known nor be easily calculable.

We propose to use the calculus method in slightly more complicated cases in a later section (§ 47). We give now two examples involving the position of the centre of pressure of the immersed rectangle, found in the preceding sections.

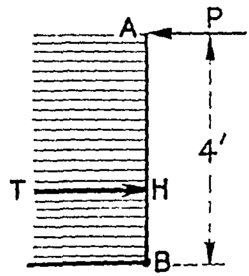


Fig. 64.

37. Example 1.—*One end of a trough of rectangular cross-section* 3 *ft. wide by* 4 *ft. deep is hinged along its lower edge and is kept in position by a horizontal force* P *lb. wt. applied to the mid-point of its upper edge. If the trough is just filled with water, find the least value of* P *necessary to prevent water flowing out.* (*The density of water* = 62·5 *lb. per cubic foot.*) (*H.S.C.*)

Let AB (Fig. 64) represent a section of the end of the trough, with A at the surface and B at the line of hinges. Then if H is the centre of pressure of the rectangular end

$$AH = \tfrac{2}{3}AB$$
$$= \tfrac{2}{3} \times 4 \text{ ft.};$$
$$\therefore \quad BH = \tfrac{1}{3} \times 4 \text{ ft.}$$

The centre of *gravity* of the rectangular end is at a depth of 2 ft., therefore, total thrust on end (T) $= 3 \times 4 \times 2 \times 62\frac{1}{2}$ lb. wt.

Taking moments about B, the least force P is given by

$$P.AB = T.BH;$$

$$\therefore \quad P \times 4 = 3 \times 4 \times 2 \times 62\tfrac{1}{2} \times \tfrac{4}{3},$$

giving $\quad P = 500 \text{ lb. wt.}$

Example 2 —*A lock gate is 8 ft wide The water on one side has a depth of 12 ft and on the other side a depth of 7 ft Find the fluid thrust and centre of pressure for each side, and hence find the magnitude and line of action of the resultant force on the gate*

The thrust on the 12 ft deep side

$$= \text{area} \times w \times \text{depth of C G of area}$$
$$= 96 \times w \times 6 = 576w$$
$$= 576000 \text{ oz wt}$$

and acts in the vertical median line at a depth of $\frac{2}{3} \times 12$ or 8 ft i e 4 ft from bottom of gate

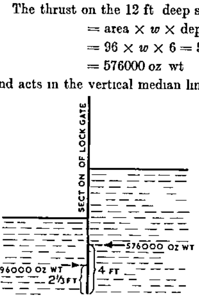


Fig 65

The thrust on the 7 ft deep side

$$= 56 \times w \times 3\tfrac{1}{2} = 196w$$
$$= 196000 \text{ oz wt},$$

and acts at a depth of $\frac{2}{3} \times 7$ or $4\frac{2}{3}$ ft i e $2\frac{1}{3}$ ft from bottom of gate

Thus forces are as in Fig 65

The resultant = (576000 − 196000) oz = 380000 oz wt and acts in the median line at a distance x from the bottom where x is given by

$$380000x = 576000 \times 4 - 196000 \times 2\tfrac{1}{3},$$
$$x = \frac{576 \times 4 - 196 \times 2\frac{1}{3}}{380} \text{ ft} = 4\ 86 \text{ ft}$$

Thus the resultant force = 380000 oz wt or 10 6 ton wt nearly, and acts at a point on the vertical median line of the gate 4 86 ft from the bottom

38 The centre of pressure on a combination of simple geometrical figures immersed vertically in a fluid

In § 28 we found that we could obtain the total thrust on an area which could be considered as the sum or difference of two or more simple geometrical shapes and we may frequently adopt exactly the same procedure in determining the position of the centre of pressure of such an area The method will be clearly illustrated by the following examples

Example 1 —*Find the position of the centre of pressure of a rectangle immersed vertically in a fluid with its upper edge horizontal and at a depth h*

Let ABCD (Fig 66) represent a rectangle immersed vertically in a fluid of density ρ, having AB (of length b) horizontal and at a depth h,

the side BC being of length a. Produce the vertical edges to meet the surface in E and F. The rectangle ABCD may then be considered as the difference between the two rectangles EFCD and EFBA, both having one edge in the surface and consequently the positions of their centres of pressure is known. It is convenient to draw up a table as follows:—

Rectangle	Area	Total Thrust	Depth of Centre of Pressure
EFCD	$b(a+h)$	$\rho . b(a+h) . \frac{1}{2}(a+h)$	$\frac{2}{3}(a+h)$
EFBA	bh	$\rho . bh . \frac{1}{2}h$	$\frac{2}{3}h$
ABCD	ab	$\rho . ab . (h + \frac{1}{2}a)$	Z

where Z denotes the required depth of the centre of pressure. Taking moments for these thrusts above the surface EF, we have

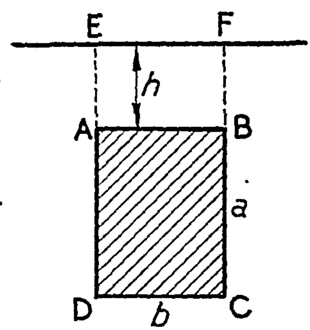


Fig. 66.

thrust on area ABCD $\times$ Z + thrust on area EFBA $\times$ its depth of C.P. =

thrust on area EFCD $\times$ its depth of C.P.

i.e.

$$\rho . ab\,(h + \tfrac{1}{2}a) \times Z + \rho . bh . \tfrac{1}{2}h \times \tfrac{2}{3}h = \rho . b(a+h) . \tfrac{1}{2}(a+h) \times \tfrac{2}{3}(a+h),$$

i.e.

$$\rho \frac{ab}{2}(2h+a)\,Z + \frac{\rho b h^3}{3} = \frac{\rho b}{3}(a+h)^3;$$

$$\therefore \quad Z = \frac{\frac{b}{3}[(a+h)^3 - h^3]}{\frac{ab}{2}(a+2h)}$$

$$= \tfrac{2}{3} . \frac{a^3 + 3a^2h + 3ah^2}{a(a+2h)}$$

$$= \tfrac{2}{3} . \frac{a^2 + 3ah + 3h^2}{a+2h}.$$

This is the depth of the centre of pressure, and on putting $h = 0$, this reduces to $Z = \frac{2}{3}a$, the result we have obtained earlier for the depth of the centre of pressure of a rectangle immersed vertically in a fluid with one edge in the surface.

Finally, the position of the centre of pressure is fixed since its depth is now known and since it must lie on the line joining the mid-point of AB to the mid-point of CD by symmetry.

Example 2 —*Find the depth of the centre of pressure of a triangle immersed vertically in a fluid with one vertex in the surface and the other two vertices at depths d_1 and d_2*

Let ABC (Fig 67) be the triangle and let CB produced cut the surface in D where $AD = a$

The triangle ABC is then the difference of two triangles, namely, ADC, ADB, both of which have an edge in the surface and consequently we may easily find the total thrusts on these triangles and the depths of their centres of pressure Thus we may obtain the total thrust on the triangle ABC by subtraction and then proceed as in the last example to find the depth of the centre of pressure of the triangle ABC Let this depth be Z and let ρ be the density of the fluid

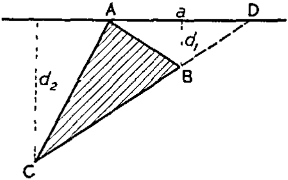


Fig 67

We draw up the following table —

TRIANGLE	AREA	TOTAL THRUST	DEPTH OF CENTRE OF PRESSURE
ADC	$\frac{1}{2}ad_2$	$\rho\ \frac{1}{2}ad_2\ \frac{1}{3}d_2$	$\frac{1}{2}d_2$
ADB	$\frac{1}{2}ad_1$	$\rho\ \frac{1}{2}ad_1\ \frac{1}{3}d_1$	$\frac{1}{2}d_1$
ABC		$\frac{a\rho}{6}(d_2^2 - d_1^2)$	Z

In this table we have not obtained the total thrust on the triangle ABC from the pressure intensity at its centre of gravity, but by subtracting the thrust on the triangle ADB from the thrust on the triangle ADC

Thus, thrust on triangle $ABC = \rho\ \frac{1}{2}ad_2\ \frac{1}{3}d_2 - \rho\ \frac{1}{2}ad_1\ \frac{1}{3}d_1$

$$= \frac{a\rho}{6}(d_2^2 - d_1^2)$$

If we now take moments for these total thrusts about the surface AD we have

$$\begin{array}{l}\text{thrust on } \triangle\ ABC \times Z + \\ \text{thrust on } \triangle\ ADB \times \tfrac{1}{2}d_1\end{array} = \text{thrust on } \triangle\ ADC \times \tfrac{1}{2}d_2;$$

$$\therefore\quad \frac{a\rho}{6}(d_2^2 - d_1^2)\,Z + \frac{a\rho}{6}d_1^2 \times \tfrac{1}{2}d_1 = \frac{a\rho}{6}d_2^2 \times \tfrac{1}{2}d_2;$$

$$\therefore\quad 2\,(d_2^2 - d_1^2)\,Z + d_1^3 = d_2^3;$$

$$\therefore\quad Z = \frac{d_2^3 - d_1^3}{2\,(d_2^2 - d_1^2)}$$

$$= \frac{d_2^2 + d_2 d_1 + d_1^2}{2\,(d_2 + d_1)}.$$

From this result we may deduce the position of the centre of pressure of a triangle immersed vertically in a fluid with one vertex in the surface and the opposite side horizontal at a depth h.

Putting $d_2 = d_1 = h$ in the above result,

$$Z = \frac{h^2 + h^2 + h^2}{2\,(h + h)}$$

$$= \frac{3h^2}{4h}$$

$$= \tfrac{3}{4}h.$$

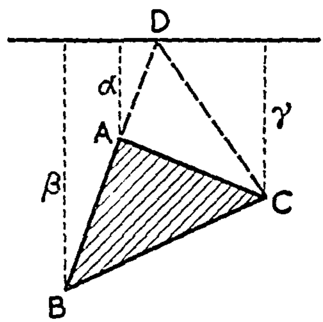


Fig. 68.

Example 3.—*Find the depth of the centre of pressure of any triangle immersed vertically in a fluid.*

Let ABC be the triangle (Fig. 68). Produce BA to meet the surface in D. Join CD.

Then if α, β, γ are the depths of A, B, C respectively, we see from the last example that the centre of pressure of the $\triangle$ DBC is at a depth $(\beta^2 + \beta\gamma + \gamma^2)/2\,(\beta + \gamma)$ and the centre of pressure of the $\triangle$ DAC is at a depth $(\alpha^2 + \alpha\gamma + \gamma^2)/2\,(\alpha + \gamma)$.

Also depth of C.G. of $\triangle$ DBC $= \frac{1}{3}$ of sum of depths of the points D, B, C $= \dfrac{\beta + \gamma}{3}$;

$$\therefore\quad \frac{\text{the thrust on } \triangle\ DBC}{\text{the thrust on } \triangle\ DAC} = \frac{\triangle\ DBC \times w \times \dfrac{\beta + \gamma}{3}}{\triangle\ DAC \times w \times \dfrac{\alpha + \gamma}{3}} \qquad (\S\ 26)$$

$$= \frac{DB \times (\beta + \gamma)}{DA \times (\alpha + \gamma)} = \frac{\beta\,(\beta + \gamma)}{\alpha\,(\alpha + \gamma)};$$

$\therefore$ depth of centre of pressure on $\triangle$ ABC

$$= \frac{\beta(\beta+\gamma) \times \dfrac{\beta^2+\beta\gamma+\gamma^2}{2(\beta+\gamma)} - a(a+\gamma)\dfrac{a^2+a\gamma+\gamma^2}{2(a+\gamma)}}{\beta(\beta+\gamma) \qquad - a(a+\gamma)}$$

(see *Tutorial Statics*, § 84)

$$= \frac{\beta(\beta^2+\beta\gamma+\gamma^2) - a(a^2+a\gamma+\gamma^2)}{2\{\beta(\beta+\gamma) - a(a+\gamma)\}}$$

$$= \frac{(\beta^3-a^3)+(\beta^2-a^2)\gamma+(\beta-a)\gamma^2}{2\{(\beta^2-a^2)+(\beta-a)\gamma\}}$$

$$= \frac{(\beta^2+a\beta+a^2)+(\beta+a)\gamma+\gamma}{2\{(\beta+a)+\gamma\}}, \quad \text{dividing by } (\beta-a)$$

$$= \frac{a^2+\beta^2+\gamma^2+a\beta+a\gamma+\beta\gamma}{2(a+\beta+\gamma)}$$

Example 4 —*Find the position of the centre of pressure of a square immersed in a fluid with one corner in the surface and a diagonal vertical*

Let ABCD (Fig 69) represent the square with AC (of length h) vertical Produce CD, CB to meet the surface in E, F respectively Then the square ABCD is the difference between the triangle CEF and the two small equal triangles ADE, ABF Since all these triangles have a side in the surface we may easily write down the thrusts on them and the depths of the centre of pressure of each of them From the geometry of the figure AF = AE = h and B, D are each at a depth of $\frac{1}{2}h$

Thus we have the following table —

TRIANGLE	AREA	THRUST	DEPTH OF CENTRE OF PRESSURE
CEF	h^2	$\rho\, h^2 \dfrac{h}{3}$	$\frac{1}{2}h$
ADE	$\frac{1}{4}h^2$	$\rho\, \frac{1}{4}h^2\, \frac{1}{3}\left(\dfrac{h}{2}\right)$	$\frac{1}{4}h$
ABF	$\frac{1}{4}h^2$	$\rho\, \frac{1}{4}h^2\, \frac{1}{3}\left(\dfrac{h}{2}\right)$	$\frac{1}{4}h$

Since

area of square ABCD = area $\triangle$ CEF − area $\triangle$ ADE − area $\triangle$ ABF,

we have

$$\text{thrust on square ABCD} = \text{thrust on } \triangle \text{ CEF} - \text{thrust on } \triangle \text{ ADE} - \text{thrust on } \triangle \text{ ABF},$$
$$= \tfrac{1}{3}\rho h^3 - \tfrac{1}{24}\rho h^3 - \tfrac{1}{24}\rho h^3,$$
$$= \tfrac{1}{4}\rho h^3.$$

If the depth of the centre of pressure of the square be Z, we have, by taking the moments of the thrusts about the surface,

$$\tfrac{1}{4}\rho h^3 \times Z = \tfrac{1}{3}\rho h^3 \times \tfrac{1}{2}h - \tfrac{1}{24}\rho h^3 \times \tfrac{1}{4}h - \tfrac{1}{24}\rho h^3 \times \tfrac{1}{4}h$$
$$= \tfrac{1}{6}\rho h^4 - \tfrac{1}{48}\rho h^4$$
$$= \tfrac{7}{48}\rho h^4;$$
$$\therefore \quad Z = \tfrac{7}{12}h,$$

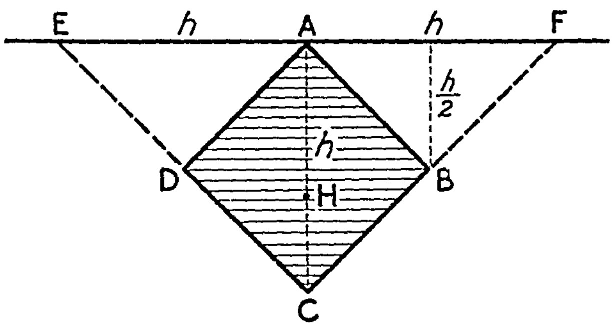


Fig. 69.

i.e. the centre of pressure of the square is at H, where AH $= \tfrac{7}{12}$AC, or if a is the length of a side of the square

$$\text{AH} = \tfrac{7}{12} . 2a \cos 45$$
$$= \frac{7a\sqrt{2}}{12}.$$

39. For an area immersed at any angle in a fluid, the position of the centre of pressure relative to the area is unchanged by rotating the area about the line of intersection of its plane with the surface.

In all our considerations of the position of centres of pressure so far, we have always stipulated that the area be immersed vertically and we indicated earlier that this involved no loss of generality. We will now establish the theorem which confirms this statement.

Consider § 33 again and let Fig. 70 represent a sectional or "end-on" view of the area in Fig. 56. Let AB be the "end-on" elevation of the plane area considered in that section, and suppose that BA produced meets the surface in O. The total thrust (T) on the area acts at H

so that H is the position of the centre of pressure Now suppose that the vertical line OAB rotates through an angle θ degrees into the position OA'B' The plane of our immersed area is now at an angle θ to the vertical, and if H' is the new position of the centre of pressure we will prove that H rotates into H', i e that the position of the centre of pressure of the area is unchanged *relative to the area*

The typical element of area a_1, which was at a depth z_1, is at a depth $z_1 \cos\theta$ after rotation, so that

$$\text{thrust on element} = \rho a_1 z_1 \cos\theta, \text{ approximately}$$

Therefore, with the notation of § 33,

$$\begin{aligned}\text{total thrust on area} &= \text{limit of } \Sigma_{r=1}^{n} \rho a_r z_r \cos\theta \\ &= \text{limit of } \rho\cos\theta\ \Sigma_{r=1}^{n} a_r z_r \\ &= \rho\cos\theta \times \text{limit of } \Sigma_{r=1}^{n} a_r z_r \qquad \text{(i)}\end{aligned}$$

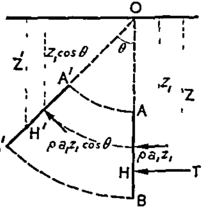


Fig 70

But the moment of the thrust on the element about the surface

$$= \rho a_1 z_1 \cos\theta\ z_1 \cos\theta \text{ approximately,}$$

so that the total moment about the surface

$$\begin{aligned}&= \text{limit of } \Sigma_{r=1}^{n} \rho a_r z_r^2 \cos^2\theta \\ &= \text{limit of } \rho\cos^2\theta\ \Sigma_{r=1}^{n} a_r z_r^2 \\ &= \rho\cos^2\theta \times \text{limit of } \Sigma_{r=1}^{n} a_r z_r^2 \qquad \text{(ii)}\end{aligned}$$

since ρ and θ are constant for all elements, and where the limits are taken as $n \to \infty$ but subject to

$$a_1 + a_2 + a_3 + \quad + a_n = \text{whole area as in § 33}$$

If the depth of the centre of pressure (H′) in the new position is Z' we have, from (i) and (ii),

$$Z' \times \text{total thrust on area} = \text{total moment of thrust on area,}$$

i.e. $Z' \times \rho \cos\theta \times \text{limit of } \Sigma a_r z_r = \rho \cos^2\theta \times \text{limit of } \Sigma a_r z_r^2$;

$$\therefore \quad Z' = \cos\theta \times \frac{\text{limit of } \Sigma a_r z_r^2}{\text{limit of } \Sigma a_r z_r}$$

$$= \cos\theta \times Z, \text{ from § 33.}$$

But $$Z' = \text{OH}' \times \cos\theta;$$

$$\therefore \quad \text{OH}' = Z = \text{OH};$$

$\therefore$ H rotates to H′, so that the position of the centre of pressure *relative to the area* is unaltered by the rotation.

Example 1.—*A rectangle is immersed in a fluid with two edges horizontal and at depths 2 ft. and 7 ft. Find the depth of the centre of pressure of the rectangle if its plane is inclined at 30° to the horizontal.*

Let AB (Fig. 71) represent a section of the rectangle and let BA produced meet the surface in O. The centre of pressure of the rectangle is at H.

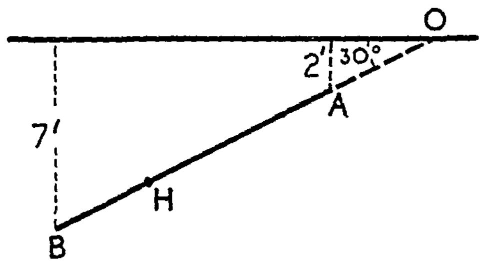


Fig. 71.

Then, for Example 1, § 38, if AB were rotated about O into a vertical line,

$$\text{OH} = \tfrac{2}{3}.\frac{a^2 + 3ah + 3h^2}{a + 2h} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where $a = \text{AB}$, $h = \text{OA}$.

In this example $\text{OA} \cos 60° = 2$;

$$\therefore \quad \text{OA} = 4 \text{ ft.} = h,$$

and $$\text{AB} = 10 \text{ ft.} = a.$$

Substituting these values in (i),

$$\text{OH} = \tfrac{2}{3}.\frac{100 + 120 + 48}{10 + 8}$$

$$= \tfrac{268}{27} \text{ ft.}$$

Therefore

$$\text{depth of H below surface} = \text{OH} \cos 60^\circ$$
$$= \tfrac{268}{27} \times \tfrac{1}{2} \text{ ft}$$
$$= 4\ 96 \text{ ft}$$

40 Centre of pressure of area immersed in several fluids which do not mix

In § 33 where we discussed methods of finding the position of the centre of pressure of a plane area, we assumed that the fluid was homogeneous If we are dealing with an area immersed in several different layers of fluids with different densities, exactly the same fundamental method is applicable, but care has to be taken in finding the pressure intensity at any particular depth For instance, if a layer of density ρ_1 and depth d rests on another liquid of density ρ_2 ($> \rho_1$) then the pressure intensity due to the liquids alone at a depth of z below the common surface is

$$\rho_1 d + \rho_2 z$$

Consequently we may, as in § 33, find the thrust on an element and hence the total thrust on the area, the moment about the surface of the thrust on the element and hence the total moment, so that we may find the depth of the centre of pressure by using the moment equation

An alternative method which may sometimes be quicker is to suppose that all the liquids have the same density, but that the thickness of each individual layer is altered so that the pressure intensity at every point of the area is the same as it is in fact Thus, in the example above, if we suppose the upper layer of liquid also to be of density ρ_2 then its depth (h) must be such that

$$\rho_2 h = \rho_1 d,$$

since $\rho_1 d$ is the pressure intensity due to the liquids alone at the common surface

Therefore $$h = \frac{\rho_1}{\rho_2} d$$

We propose to work through an example by both methods for the sake of comparison

Example 1 —*A layer of liquid of density ρ and depth a, is superimposed on a layer of liquid of density 2ρ and depth $> 2a$ A square lamina of side $2a$ is immersed vertically with its upper edge in the surface common to the two liquids Find the centre of liquid pressure below the upper edge of the lamina* (*Inter Sc*)

Method (i) —The square lamina is represented in Fig 72 Consider a horizontal element at a depth z below the common surface

of width δz. Then the pressure intensity at this depth is $\rho a + 2\rho z$ and consequently we have

$$\text{thrust on element} = 2a\delta z\,(\rho a + 2\rho z);$$

$\therefore$ moment about free surface of thrust on element

$$= 2a\delta z\,(\rho a + 2\rho z)(a + z)$$

since $a + z$ is the depth of the element below the free surface;

$\therefore$ total moment about free surface

$$= \int_0^{2a} 2a\,(\rho a + 2\rho z)(a + z)\,dz,$$

where the range of values for z (which is measured from the common surface) is O to $2a$.

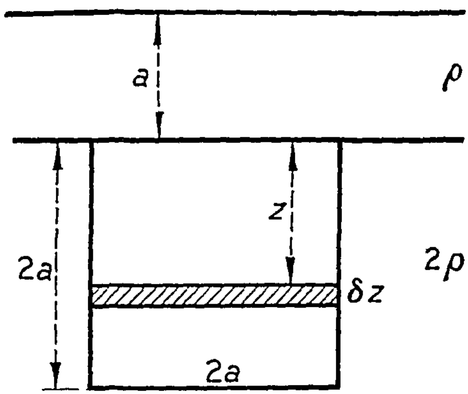


Fig. 72.

Thus

$$\text{total moment about free surface} = 2a\rho\int_0^{2a}(a + z)(a + 2z)\,dz$$

$$= 2a\rho\int_0^{2a}(a^2 + 3az + 2z^2)\,dz$$

$$= 2a\rho\left[a^2z + \frac{3az^2}{2} + \frac{2z^3}{3}\right]_0^{2a}$$

$$= 2a\rho\,[2a^3 + 6a^3 + \tfrac{16}{3}a^3]$$

$$= \tfrac{80}{3}a^4\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also, since the pressure intensity at the centre of gravity of the square lamina $= \rho a + 2\rho . a$

$$= 3a\rho,$$

we have

total thrust on square lamina $= (2a)^2.3a\rho$

$$= 12a^3\rho.$$

Thus if Z is the depth below the common surface of the centre of pressure of the square lamina, the moment of the total thrust about the free surface

$$= 12a^3\rho\,(Z + a) \;.. \;\;.. \;... \;\;.. \quad \text{(ii)}$$

and equating this to (i),

$$12a^3\rho\,(Z + a) = \tfrac{80}{3}a^4\rho,$$

$$\therefore \; Z + a = \tfrac{20}{9}a$$

$$Z = \frac{20a}{9} - a$$

$$= \frac{11a}{9}$$

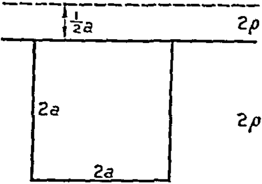


Fig. 73.

Method (ii).—The pressure intensity in the common surface is $a\rho$, and so we may theoretically replace the upper liquid by one of density 2ρ, providing its thickness is $\frac{1}{2}a$ since then the pressure intensity in the common surface, i.e. $\frac{1}{2}a\,(2\rho)$, is still equal to $a\rho$ (Fig. 73). Thus our problem reduces to finding the centre of pressure of a square immersed vertically in a homogeneous liquid of density 2ρ with its upper edge horizontal and at a depth $\frac{1}{2}a$ from the theoretical free surface. This is a particular case of the rectangle, vertical edge a, upper edge at depth h, which we solved in § 38 (Example 1), when we found

$$\text{depth of C. of P. below free surface} = \tfrac{2}{3}\cdot\frac{a^2 + 3ah + 3h^2}{a + 2h} \;.\;.\; \text{(iii)}$$

To use this result to find the depth of the centre of pressure of the square lamina we replace a by $2a$ and h by $\frac{1}{2}a$. Then (iii) becomes

$$\begin{array}{l}\text{depth of C. of P. below} \\ \textit{theoretical}\text{ free surface}\end{array} = \frac{2}{3}\cdot\frac{4a^2 + 6a\,.\,\frac{1}{2}a + \frac{3}{4}a^2}{2a + a}$$

$$= \frac{31a}{18}.$$

Thus, the depth of the centre of pressure of the square lamina below the common surface

$$= \frac{31a}{18} - \frac{a}{2}$$

$$= \frac{11a}{9}.$$

Exercises V

Take the density of water as 62·5 lb. per cub. ft.

(1) A water tank has a rectangular section of height 4 ft. and width 6 ft. Find the total liquid thrust on an end when the tank contains water to a depth of 3 ft., and the height of the centre of pressure above the base.

(2) A right-angled triangle of sides 3, 4 and 5 ft. is immersed vertically in a liquid and has its hypotenuse in the surface. Find the depth of the centre of liquid pressure.

(3) If, in the last question, the triangle is immersed so that its plane makes an angle of 10° with the surface instead of being vertical, find the depth of the centre of liquid pressure.

(4) A triangle ABC, right-angled at A, is immersed vertically in a liquid of density 90 lb. per cub. ft., with AB in the surface. If AB = 2 ft., AC = 4 ft., find the total liquid thrust on the triangle and the distances of the centre of pressure (H) from A, B and C.

(5) A lock-gate, 14 ft. high and 10 ft. wide, has water to a height of 12 ft. on one side of it and to a height of 8 ft. on the other. Find the magnitude of the resultant thrust on it and the position of the centre of pressure.

(6) The depths of the water on the two sides of a lock-gate are 12 ft. and 4 ft. Find the resultant force on the gate due to the pressure of the water, and prove that it acts at a point whose height above the bottom of the lock is 52 in. Take the breadth of the gate to be 15 ft., and 1 cub. ft. of water to weigh 62·5 lb. (Inter. Sc.)

(7) The vertical side of a tank of water contains a square trap door, edge 2 ft., which is hinged along its lower horizontal edge, and is kept from opening outwards by a string attached to its upper edge, which can support a maximum horizontal thrust of 18 lb. wt. Prove that the trap-door will begin to open when $\frac{3}{5}$ of its area is covered. (Inter. Sc.)

(8) A rectangular lamina of sides a, $2a$, is immersed vertically in a liquid with the side of length a in the surface. Show that the distance between the centres of pressure of the two triangles into which the lamina is divided by a diagonal is $5a/8$.

(9) A square lamina ABCD of side $2a$ is immersed vertically in liquid, with AB in the surface. A triangular portion of height a, whose base of length a lies in AB, is removed. Find the depth of the centre of pressure of the remaining portion. (Inter Sc)

(10) A rectangle ABCD of height 6 ft is immersed vertically in water with the edge AB in the water surface. The diagonals AC, BD intersect at O. Show that the centre of pressure of the triangle AOD is at a depth of $3\frac{1}{2}$ ft. (Inter Sc)

(11) Two opposite edges of a rectangle are horizontal and at depths h, k below the surface of a liquid, prove by direct integration that the depth of the centre of pressure is

$$\tfrac{2}{3}(h^2 + hk + k^2)/(h + k)$$

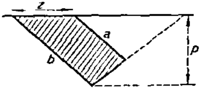
(12) A rectangular area is immersed vertically in water with one side horizontal at a depth of 9 ft, and the opposite side at a depth of 15 ft. Show that the centre of pressure is 3 in below the middle point of the rectangle.

(13) A flat-bottomed tank is divided into two compartments by a vertical door of width 3 m. Find the magnitude, direction and position of the resultant hydrostatic thrust on the door when one compartment is filled with oil of density 1.5 grm per c.cm. to a depth of 10 m., and the other is filled with a different oil of density 2·0 grm per c cm to a depth of 20 m.

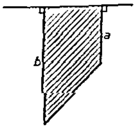
(14) A triangle ABC, right-angled at A, is immersed vertically in a liquid with A in the surface and BC horizontal. If the side AB (of length a) is inclined to the horizontal at an angle θ, find the perpendicular distances of the centre of pressure from the three sides.

(15) A square lamina is partly immersed in a uniform liquid with its plane and one diagonal vertical. The length of the diagonal is $2a$ and the depth below the surface of the lowest corner is $3a/2$. Find the depth of the centre of pressure of the part immersed. (Inter Sc)

(16) In both the cases below, the lamina is immersed vertically in a liquid of uniform density. Find the depth of the centre of pressure in terms of the given quantities.



(i)



(ii)

(17) A lamina in the form of an equilateral triangle is immersed vertically in water with one side in the free surface.

Prove that the thrusts, due to the water pressure on the two portions into which one face is divided by a horizontal line drawn on that face through the centre of pressure, are equal. (Inter Sc)

(18) Find, by direct integration, the depth of the centre of pressure in each of the following cases:—

(i) A rectangle of sides a, b immersed vertically in a fluid with its upper edge (b) horizontal and at a depth h.

(ii) A triangle ABC immersed vertically in a fluid with A in the surface and BC horizontal at a depth h.

(iii) A triangle ABC immersed vertically in a fluid with A at a depth k and BC horizontal at depth $h + k$.

(19) Find the position of the centre of pressure of a triangle whose plane is vertical, its base being horizontal at a depth of 4 ft. below the surface and a height of 12 ft. above the opposite vertex.

(20) ABC is a triangular area immersed vertically in water with C in the surface and AB horizontal; show how to divide the area by a horizontal line, PQ, into two portions on which the pressures are equal, P and Q being points in AC and BC respectively.

If h is the length of the perpendicular from C on AB, prove that the height above AB of the centre of pressure on the area APQB in the above case is $\frac{1}{8}h\,(3 \times 4^{\frac{1}{3}} - 4)$.

(21) Show that the depth of the centre of pressure of a rhombus totally immersed in a homogeneous liquid with one diagonal vertical and its centre at a depth h is $(d^2 + 24h^2)/24h$, where d is the length of the vertical diagonal.

(22) A square, with sides of length a, is immersed vertically in water with its centre at a depth h ($> a/\sqrt{2}$). If the square is rotated in a vertical plane about its centre, show that the depth of the centre of pressure is constant and equal to $(a^2 + 12h^2)/12h$.

(23) Find the depth of the centre of pressure of a rectangle immersed vertically in a liquid whose density varies as the depth below the surface, the upper edge of the rectangle being in the surface and the lower edge at a depth a.

(24) A cubical box, whose inner edges are 1 ft. in length, is standing on a horizontal base and is half filled with water and half with mercury of specific gravity 13·6. Find the total fluid thrust in lb. wt. on a vertical face of the box and the depth of the centre of pressure on this face.

(25) A parallelogram is immersed vertically in a liquid whose density varies as the depth below the surface, with two of its sides horizontal at depths a and b respectively. Show that the depth of the centre of pressure is $3\,(b^4 - a^4)/4\,(b^3 - a^3)$.

ANSWERS

1. 1687·5 lb. wt., 1 ft. 2. 1·2 ft. 3. $2\frac{1}{4}$ in.

4. 480 lb. wt., AH = CH = $\sqrt{5}$ ft., BH = $2\frac{1}{2}$ ft.

5. 25,000 lb. wt., 5·067 ft. above bottom of gate.

6. 60,000 lb. wt. 9. $63a/46$.

Again, if $\bar{z}$ denotes the depth of the centre of gravity of the area below the surface, we have from the definition of the centre of gravity

$$A\bar{z} = \lim_{n \to \infty} \Sigma_1^n a_r z_r \tag{iii}$$

Using relations (ii) and (iii) in (i), we obtain

$$Z = \frac{Ak_s^2}{A\bar{z}},$$

$$= \frac{k_s^2}{\bar{z}} \tag{iv}$$

so that the depth of the centre of pressure below the surface is the square of the radius of gyration of the area about the surface divided by the depth of the centre of gravity below the surface

By the theorem of parallel axes (*Tutorial Dynamics*, § 268) we may obtain this result in a more convenient form by expressing the radius of gyration about the surface line SS (Fig 74) in terms of the

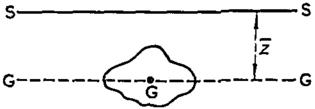


Fig 74

radius of gyration about a line GG through the centre of gravity of the area parallel to the line of intersection of the area and the surface and the depth of the C G

This theorem states that

$$k_s^2 = k_g^2 + \bar{z}^2 \tag{v}$$

where k_g is the radius of gyration of the area about the axis GG through the centre of gravity

Thus (iv) becomes

$$Z = \frac{k_g^2 + \bar{z}^2}{\bar{z}}$$

$$= \frac{k_g^2}{\bar{z}} + \bar{z} \tag{vi}$$

Consequently, if we know the depth of the centre of gravity of an area below the surface and also its radius of gyration about a horizontal axis through the centre of gravity in the plane of the figure,

then we may write down immediately the depth of the centre of pressure.

It may here be noticed that the position of the centre of pressure coincides with the *centre of oscillation* (see *Tutorial Dynamics*, §§ 286, 287), *i.e.* if the body were oscillated about a horizontal line in the surface perpendicular to the plane of the paper, then the depth of the centre of gravity is equal to the length of the simple equivalent pendulum.

Example 1.—*Find the depth of the centre of pressure of a rectangle immersed vertically with two edges horizontal, the uppermost being at a depth h and having a vertical edge of length a.*

For this rectangle

$$k_g^2 = \frac{a^2}{12}, \qquad \textit{(Tutorial Dynamics, § 270)}$$

and

$$\bar{z} = h + \tfrac{1}{2}a.$$

Therefore

$$Z = \frac{a^2}{12\,(h + \frac{1}{2}a)} + h + \tfrac{1}{2}a$$

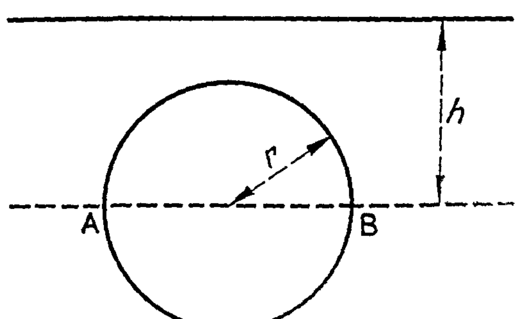
$$= \frac{a^2 + 3\,(a + 2h)^2}{6\,(a + 2h)}$$

$$= \frac{4a^2 + 12ah + 12h^2}{6\,(a + 2h)}$$

$$= \tfrac{2}{3} \cdot \frac{a^2 + 3ah + 3h^2}{a + 2h}.$$

We obtained this result in § 38, Example 1.

Example 2.—*Find the depth of the centre of pressure of a circle immersed vertically in a liquid.*



The radius of gyration about the diameter AB is $\frac{1}{2}r$ (*Tutorial Dynamics*, § 276),

$$\therefore\ k_g^2 = \frac{r^2}{4},$$

and the depth of the centre of gravity of the circle (i e the centre) is h

Thus $$Z = \frac{r^2}{4h} + h,$$

and for the case when the circumference of the circle touches the surface, $h = r$ and

$$Z = r + \tfrac{1}{4}r$$
$$= \frac{5r}{4}$$

Example 3—*A triangle ABC is immersed in a liquid with AB in the surface and CD inclined at 60° to the vertical, where D is the foot of the perpendicular from C on to AB Find the depth of the centre of pressure*

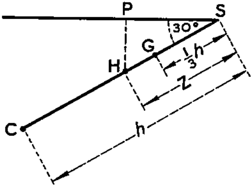


Fig 76

Let CD $= h$ Fig 76 represents an end view, G and H being the positions of the centre of gravity and centre of pressure respectively S is the end view of the line of intersection of the plane of the triangle and the surface

Thus the distance of G from S is $\frac{1}{3}h$ and if we now denote the distance of H from S by Z we have

$$Z = \frac{k_s^2}{\bar{z}},$$

where $\bar{z}$ is measured along the plane of the triangle (see § 39), i e $\bar{z} = \frac{1}{3}h$

The radius of gyration of the triangle about the side CD is $\frac{h^2}{6}$ (see *Tutorial Dynamics*, § 274), so that

$$Z = \frac{h^2}{6\,\dfrac{h}{3}}$$
$$= \tfrac{1}{2}h$$

Thus PH (the depth of the centre of pressure)

$$= Z \cos 60^\circ \qquad (\text{by } \S\, 39)$$
$$= \tfrac{1}{2}h \,.\, \tfrac{1}{2}$$
$$= \tfrac{1}{4}h.$$

43. The depth of the centre of pressure of any plane area immersed in a fluid is greater than the depth of the centre of gravity provided the area is not horizontal

If the area is immersed vertically, then with the notation we have used previously, Z denotes the depth of the centre of pressure and $\bar{z}$ the depth of the centre of gravity.

If the area is immersed at an angle, we let Z, $\bar{z}$ denote the distances of the centre of pressure and centre of gravity respectively from the line of intersection of the plane of the figure and the surface (see § 39).

If the area is horizontal, then since both the centre of pressure and centre of gravity are points on the area, they are both at the same depth.

From the last section we have [equation (vi)]

$$Z = \frac{k_g^2}{\bar{z}} + \bar{z},$$

where k_g is the radius of gyration of the figure about a horizontal line through its centre of gravity.

Thus $$Z - \bar{z} = \frac{k_g^2}{\bar{z}}.$$

Since k_g^2 is essentially positive, $\dfrac{k_g^2}{\bar{z}} > 0$, so that

$$Z > \bar{z}.$$

We notice that since k_g^2 has a constant value for any particular figure,

$$\frac{k_g^2}{\bar{z}} \to 0 \text{ as } \bar{z} \to \infty,$$

i.e. the distance between the centres of gravity and pressure becomes zero only at an infinite depth.

Example 1.—*A rectangle is immersed vertically in a fluid with two of its sides horizontal, the uppermost being at a depth h. Show that the centre of pressure is at a greater depth than the centre of gravity and find the value of h for which the difference between the depths of the centre*

of pressure and centre of gravity is $\frac{1}{100}a$, *where* a *is the length of the vertical side of the rectangle*

The depth of the centre of gravity is $h + \frac{a}{2}$

The depth of the centre of pressure is $\frac{2}{3}\frac{(a^2 + 3ah + 3h^2)}{a + 2h}$, see § 38, Example 1

Therefore, the depth of the centre of pressure is greater than the depth of the centre of gravity if

$$\frac{2}{3}\frac{(a^2 + 3ah + 3h^2)}{a + 2h} > h + \frac{a}{2},$$

i e if $\quad 4(a^2 + 3ah + 3h^2) > 3(a + 2h)^2,$

i e if $\quad 4a^2 + 12ah + 12h^2 > 3a^2 + 12ah + 12h^2,$

i e if $\quad a^2 > 0,$

which is always true

Thus, the difference between the depths of the centres of pressure and gravity

$$= \frac{2}{3}\frac{(a^2 + 3ah + 3h^2)}{a + 2h} - \left(h + \frac{a}{2}\right)$$

$$= \frac{a^2}{6(a + 2h)},$$

when $h = 0$, this difference $= \frac{a}{6}$ (which is clear independently, since then a side of the rectangle is in the surface and the difference is $\frac{2}{3}a - \frac{1}{2}a = \frac{a}{6}$)

As $h \to \infty$, $\frac{a^2}{6(a + 2h)} \to 0$ so that the greater the depth of immersion, the more nearly do the centres of gravity and pressure coincide

When $\quad \frac{a^2}{6(a + 2h)} = \frac{1}{100}a,$

$$100a^2 = 6a^2 + 12ah,$$

or $\quad 94a = 12h,$

and $\quad h = \frac{94}{12}a$

$$= 7\ 83a,$$

i e the upper edge is at a depth of 7 83a before the difference in depth of the centres of pressure and gravity is $\frac{1}{100}a$

44. Graphical determination of centres of pressure

In Chapter IV (§ 32) we considered a graphical method of finding the total thrust on an immersed area. We may extend this method to find the depth of the centre of pressure.

We merely require to find the area of the *second* derived figure, which is the derived figure of the first derived figure (Fig. 77). Consider the typical element CD of the first derived figure. Draw perpendiculars from C and D on to the datum line XX and join the feet of these perpendiculars to O and let these lines cut off a length c_r between C and D.

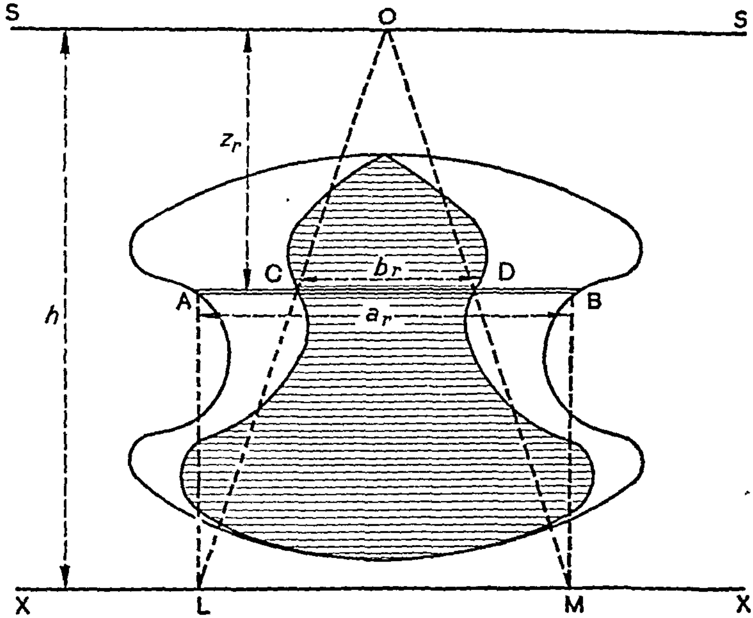


Fig. 77.

Then, as in the case of the first derived figure, we shall have, from the similar triangles so formed,

$$\frac{c_r}{z_r} = \frac{b_r}{h}$$

or

$$b_r z_r = h c_r \quad \text{.......................} \quad \text{(i)}$$

The depth (Z) of the centre of pressure below the surface is given by

$$Z = \frac{\lim\limits_{n \to \infty} \Sigma_{r=1}^{n} \rho a_r l z_r^2}{\lim\limits_{n \to \infty} \Sigma_{r=1}^{n} \rho a_r l z_r}$$

$$= \frac{\lim_{n\to\infty} \Sigma_{r=1}^{n} \rho a_r l z_r^2}{\text{total thrust}}$$

$$= \frac{\lim_{n\to\infty} \Sigma_{r=1}^{n} \rho a_r l z_r^2}{\rho A_1 h} \tag{ii}$$

from § 32

When finding the first derived figure we had the relation for the typical strip

$$a_r z_r = h b_r$$
$$\rho l a_r z_r^2 = \rho l h b_r z_r$$
$$= \rho l h^2 c_r$$

from (i)

$$\lim_{n\to\infty} \Sigma_{r=1}^{n} \rho l a_r z_r^2 = \lim_{n\to\infty} \Sigma_{r=1}^{n} \rho l h^2 c_r$$
$$= \rho h^2 \lim_{n\to\infty} \Sigma_{r=1}^{n} l c_r$$
$$= \rho h^2 A_2$$

where A_2 is the area of the second derived figure so that

$$A_2 = \lim_{n\to\infty} \Sigma_{r=1}^{n} l c_r$$

Using this relation in (ii)

$$Z = \frac{\rho A_2 h^2}{\rho A_1 l}$$
$$= \frac{A_2}{A_1} h$$
$$= \frac{\text{area second derived figure}}{\text{area first derived figure}} \times h$$

Notice that from the note at the end of § 32 this means that the position of the centre of pressure of the original figure is the same as the centre of gravity of the first derived figure

Example 1—*To use the above method to determine the depth of the centre of pressure of a rectangle immersed vertically in a fluid with one side in the surface*

In general the first and second derived figures are irregular and the usefulness of the above methods depends on being able to find their areas practically as they cannot be found by calculation In the

case of a rectangle, however, the first derived figure will be straight-sided and we may confirm our results as follows.

Let ABCD (Fig. 78) denote the rectangle; then if O is the pole chosen, the triangle ODC will be the first derived figure, so that

$$\begin{array}{c}\text{depth of centre of pressure}\\ \text{of rectangle ABCD}\end{array} = \begin{array}{c}\text{depth of centre of gravity}\\ \text{of } \triangle \text{ ODC}\end{array}$$

$$= \tfrac{2}{3}h,$$

where h is the length of the vertical side of the rectangle.

We have obtained this same result earlier by other methods.

45. Effective surface

In the last two chapters we have been primarily concerned with the total thrust and centre of pressure of an immersed plane area due to the liquid alone and consequently we have omitted the atmospheric pressure from the expression for the pressure intensity at a depth in the liquid, taking simply

pressure intensity at a depth z due to the liquid alone $= \rho z$.

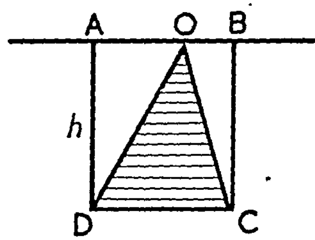


Fig. 78.

If, for any reason, the absolute total thrust on a plane area is required we need to use (see § 17)

pressure intensity at depth $z = \mathrm{P} + \rho z$,

where P is the atmospheric pressure. This may most easily be done by imagining a layer of liquid of height h superimposed on the liquid we are considering and of the same density ρ, such that the increased pressure intensity due to the superimposed liquid alone is equal to the atmospheric pressure. This will be the case when h is given by $\mathrm{P} = \rho h$ and then

$$\text{pressure intensity at depth } z = \rho h + \rho z$$
$$= \rho\,(h + z).$$

This says, in effect, that the inclusion of the atmospheric pressure is the same as supposing that we had a layer of thickness h of liquid superimposed on the true surface and that at the imaginary surface of this layer there is no atmospheric pressure acting. This imaginary surface is called the *effective surface.*

Example.—*To find the height of the effective surface for water weighing* $62\frac{1}{2}$ *lb. per cub. ft. if the atmospheric pressure is* 14·7 *lb. wt. per sq. in.*

The height of the effective surface is given by

$$\rho h = 14\ 7 \text{ lb /sq in}$$
$$= 14\ 7 \times 144 \text{ lb /sq ft},$$
$$h = \frac{14\ 7 \times 144}{62\frac{1}{2}} \text{ ft}$$
$$= 33\ 87 \text{ ft}$$

The conception of an effective surface may consequently be used to find the absolute total thrust on a plane area and the position of the absolute centre of pressure as distinct from the total liquid thrust and the centre of liquid pressure

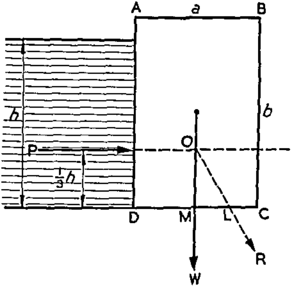


Fig 79

46 Practical applications Retaining walls and dams

A very important practical application of the knowledge we have gained of the total thrust of a liquid on an immersed area and of the position of its resultant occurs in the construction of retaining walls and dams

Let us take the simplest case of a retaining wall of rectangular section and consider the forces acting on it per unit length of wall when it keeps back a certain height of liquid From this we may determine the necessary thickness of wall in order that the wall will not fail by (a) overturning or (b) breaking

(a) *Failure by Overturning* —Let ABCD (Fig 79) represent a section of the wall, height b ft and thickness a ft, which keeps back a liquid of depth h ft

The thrust of the liquid (P) will act at $\frac{1}{3}h$ above the base and the weight of the wall (W) acts through the centre of gravity of the wall. These are both forces per unit length of the wall and it is clear that the wall will not overturn if the moment of W about C is greater than the moment of P about C, since C is the corner about which overturning would take effect.

Thus the wall will not overturn if

$$W \times \tfrac{1}{2}a > P \times \tfrac{1}{3}h \quad \text{.....................(i)}$$

If the material of the wall weighs w lb. wt. per cub. ft., then the weight (W) of 1 ft. length of wall is wab lb. wt.

If the density of the liquid is ρ lb. per cub. ft., then from the rule for the total thrust

$$P = \rho h \times \frac{h}{2} \text{ lb. wt.}$$

Substituting these values of P and W in equation (i): wall will not overturn if

$$\tfrac{1}{2}wa^2b > \tfrac{1}{6}\rho h^3,$$

i.e. if

$$a^2 > \frac{\rho h^3}{3wb}.$$

Assuming that b and h are fixed quantities, this determines the thickness a in order that the wall shall not overturn.

We may achieve this same result by the alternative method of finding the position of the line of action of the resultant of P and W. This method has advantages, as we shall see later, when considering the breaking force in the wall.

Let O (Fig. 79) be the point of intersection of the forces P and W and let the resultant (R) of these forces cut the base CD in L. Then, if M is the mid-point of CD, the triangle OLM is a triangle of forces, so that we have

$$\frac{W}{OM} = \frac{P}{LM} = \frac{R}{OL}.$$

The first two of these gives

$$LM = \frac{P}{W}.OM$$

$$= \frac{\frac{1}{2}\rho h^2}{wab}.\frac{h}{3}$$

$$= \frac{\rho h^3}{6wab}.$$

Clearly the wall will not overturn if L is to the left of C,

i e if $$LM < \frac{a}{2},$$

i e if $$\frac{a}{2} > \frac{\rho h^3}{6wab},$$

i e if $$a^2 > \frac{\rho h^3}{3wb},$$

which is the same result as we obtained before

(b) *Failure by Breaking*—In general, long before a wall would fail by overturning it would fail by crushing or rupture and care must be taken in designing a wall that the resultant of the weight of the wall and the thrust of the liquid does not act at such a point as to induce this to occur The assumption made is that a masonry wall is not expected to be able to withstand tensile stress, only compressive stress It is shown in books on hydraulics that in order that there shall be no tensile stress in the wall, the resultant must cut the base of the wall within the middle third of its thickness

Referring to Fig 79 again on this assumption in order that there shall be no tensile stress across the base of the wall,

$$\text{LM must be less than } \frac{a}{6},$$

i e $$\frac{\rho h^3}{6wab} < \frac{a}{6},$$

i e $$a^2 > \frac{\rho h^3}{wb}$$

giving a greater safe thickness than that necessary only to prevent overturning

So far we have considered the retaining wall as a whole, but it is clearly necessary to consider the stability of any portion of the wall with respect to any horizontal plane—not only the base

Consider the same wall of rectangular section, but, for simplicity, suppose the liquid rises to the top of the wall Let EF (Fig 80) be any horizontal section of the wall at a depth y below the top of the wall We may consider the forces acting on the part ABFE and find the point P, where their resultant cuts the plane EF The locus of the point P is called the *line of resistance* for the wall

The forces on the part we are considering, namely ABFE, are its weight which is $w\,ay$ and the thrust of liquid ($\frac{1}{2}\rho y^2$) which acts at a distance $\frac{1}{3}y$ above EF If K is the point of intersection of the lines

of action of these two forces and N the mid-point of EF, then, as previously, KNP is a triangle of forces and we have

$$\frac{way}{\mathrm{KN}} = \frac{\frac{1}{2}\rho y^2}{\mathrm{NP}}.$$

If we denote NP by x, then this is

$$\frac{way}{\frac{1}{3}y} = \frac{\frac{1}{2}\rho y^2}{x},$$

or

$$y^2 = \frac{6wa}{\rho}.x,$$

showing that the line of resistance in this case is a parabola, and since $y = 0$ when $x = 0$, it passes through the mid-point of AB.

If there is to be no tensile stress anywhere in the wall then for every

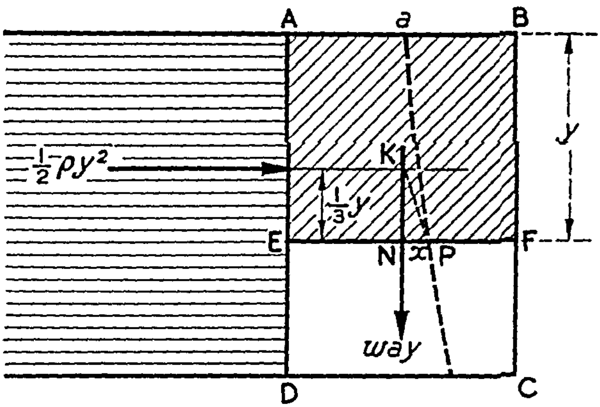


Fig. 80.

section the resultant of the forces on the wall above the section must cut the section within the middle third of its thickness, *i.e.* the line of resistance must lie wholly within the middle third section of the wall.

There are many other problems which must be taken into account when designing a wall or dam, such as the prevention or allowance for seepage of the liquid below the base of the wall with the consequent upthrust this would produce—we have simply tried to indicate in this section how some hydrostatic principles form the basis on which the practical engineer must go to work.

47. Miscellaneous Examples

We conclude this chapter with a selection of miscellaneous worked examples.

Example 1 —*A pipe of square section contains liquid of density ρ and is closed by a trap-door hinged along its upper edge and inclined at 30° to the vertical so that the liquid in the pipe tends to swing the door open Find the least weight of the door if it does not do so*

Let the side of the square be a ft and suppose that AB (Fig 81) represents the trap-door hinged along the edge at A If G and H are the centre of gravity and centre of pressure of the door respectively we have

$$AG = \tfrac{1}{2}AB$$
$$= \tfrac{1}{2}a \sec 30^\circ \quad \text{since } AB \cos 30^\circ = a$$
$$= \frac{a}{\sqrt{3}}$$

and
$$AH = \tfrac{2}{3}AB$$
$$= \frac{4a}{3\sqrt{3}}$$

If P is the total thrust of the liquid (which will act at right angles

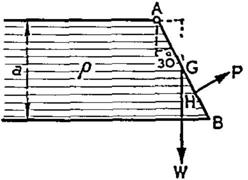


Fig 81

to the door) then the least weight (W) of the door in order not to open will be given when the moment of W about A equals the moments of P about A,

i e when $$W \cdot AG \cos 60^\circ = P \cdot AH$$

i e when $$W \cdot \frac{a}{\sqrt{3}} \cdot \tfrac{1}{2} = P \cdot \frac{4a}{3\sqrt{3}}$$

i e when $$W = \frac{8P}{3}$$

and P the total thrust equals $a \times AB \times \rho \times \tfrac{1}{2}a$

i e $$P = \tfrac{1}{2}\rho a^2 \sec 30^\circ$$
$$= \frac{\rho a^2}{\sqrt{3}}$$

i e $$W = \frac{8\rho a^3}{3\sqrt{3}} \text{ lb wt}$$

Example 2.—*To find the position of the centre of pressure of a semicircle immersed vertically in a liquid with its bounding diameter in the surface.*

Let AB (Fig. 82) represent an element of the semicircle at depth z and of thickness δz. Let its length be $2x$, *i.e.* x is measured from the axis of symmetry, somewhere on which the centre of pressure must lie, and let r be the radius of the semicircle. Then if ρ is the density of the liquid we have

$$\text{Area of elemental strip} = 2x\delta z.$$

$$\text{Thrust on strip} = 2x\delta z.\rho z.$$

Moment about surface of

$$\text{thrust on strip} = 2x\delta z.\rho z.z;$$

$$\therefore \quad \text{total moment about surface} = \int_0^r 2\rho x z^2 dz,$$

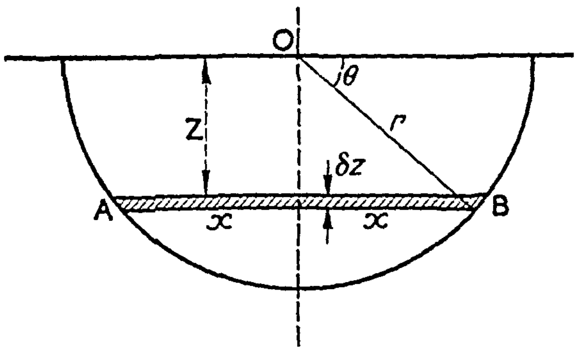


Fig. 82.

$$\textit{total moment} = 2\rho\int_0^r xz^2 dz$$

$$= 2\rho\int_0^r z^2\sqrt{(r^2 - z^2)}\,dz,$$

$$\text{since } x^2 + z^2 = r^2.$$

To integrate this expression we need to make a trigonometric transformation.

Put $$z = r\sin\theta,$$

then $$dz = r\cos\theta\, d\theta,$$

and $$r^2 - z^2 = r^2 - r^2\sin^2\theta$$

$$= r^2\cos^2\theta.$$

When we change the variable from z to θ we must also change the limits.

Thus when $z = r$ at the upper limit we have

$$r = r \sin \theta,$$

so that $\sin \theta = 1$

i e $\theta = \pi/2$

and at the lower limit when $z = 0$

then $r \sin \theta = 0,$

$$\theta = 0$$

Thus the integral for the total moment

$$= 2\rho \int_0^{\pi/2} r^2 \sin^2 \theta \; r \cos \theta \; r \cos \theta \, d\theta$$

$$= 2\rho r^4 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta$$

but $\sin^2 \theta \cos^2 \theta = \frac{1}{4} \sin^2 2\theta$

$$= \tfrac{1}{4} (\tfrac{1}{2} - \tfrac{1}{2} \cos 4\theta)$$

Thus total moment $= 2\rho r^4 \int_0^{\pi/2} \frac{1}{4} (\frac{1}{2} - \frac{1}{2} \cos 4\theta) \, d\theta$

$$= 2\rho r^4 \left[\tfrac{1}{8}\theta - \tfrac{1}{8} \frac{\sin 4\theta}{4} \right]_0^{\pi/2}$$

$$= 2\rho r^4 \, \tfrac{1}{8} \, \frac{\pi}{2}$$

$$= \frac{\pi \rho r^4}{8}$$

The total moment equals the total thrust multiplied by the depth of the centre of pressure and the total thrust in this case

$$= \frac{\pi r^2}{2} \, \rho \, \frac{4r}{3\pi}$$

$$= \tfrac{2}{3}\rho r^3 \text{ (see § 27)}$$

Thus if Z be the depth of the centre of pressure we have

$$Z \times \tfrac{2}{3}\rho r^3 = \frac{\pi \rho r^4}{8}$$

$$Z = \frac{3\pi r}{16}$$

Example 3 —*A retaining wall is to be built with dimensions as indicated in the diagram (Fig 83) to keep back oil (specific gravity 1 9) The section is a trapezium and the face of the wall in contact with the oil is vertical Find the least weight per cubic foot of the masonry if the resultant of the forces on the wall is to cut the base within its middle third*

We first require to find the distance (x) of the centre of gravity (G) of the wall from the vertical face If the masonry weighs w lb wt

per cub. ft. the weight of 1 ft. length of the wall is $60w + 15w$, *i.e.* $75w$. The rectangular part weighs $60w$ and the distance of its C.G. from the vertical face is 3 ft., while the weight of the triangular portion is $15w$ and its C.G. is at a distance of $(6 + \frac{1}{3}.3)$ ft., *i.e.* 7 ft.

Taking moments about the vertical face, to find x, we have

$$60w.3 + 15w.7 = 75wx,$$

giving
$$x = \tfrac{19}{5} \text{ ft.}$$

The total liquid thrust on the dam $= 10.5\rho$ where ρ is the density of the oil. If M is the point where the resultant of the thrust and

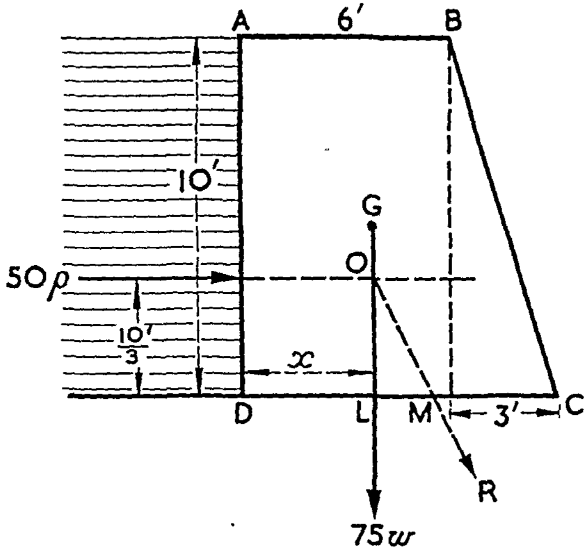


Fig. 83.

the weight cuts the base then, as before, $\triangle$ OLM is a triangle of force and we have

$$\frac{75w}{\text{OL}} = \frac{50\rho}{\text{LM}};$$

$$\therefore \quad \text{LM} = \frac{50\rho}{75w}.\text{OL}$$

$$= \frac{2\rho}{3w}.\frac{10}{3}$$

$$= \frac{20\rho}{9w};$$

$$\therefore \quad \text{DM} = \frac{19}{5} + \frac{20\rho}{9w},$$

and if M is to lie within the middle third of DC, then DM must be less than 6 ft

w must be such that $\quad DM < 6$

i e $$\frac{20\rho}{9w} + \frac{19}{5} < 6$$

i e $$\frac{20\rho}{9w} < \frac{11}{5},$$

i e $$99w > 100\rho$$

i e $$w > \tfrac{100}{99}\rho$$

i e $$> \tfrac{100}{99} \times 62\tfrac{1}{4} \times 1\ 9 \text{ lb wt per cub ft}$$

i e $$> 119\ 9 \text{ lb wt per cub ft}$$

i e the masonry must weigh at least 119 9 lb wt per cub ft

Exercises VI

(1) A triangle is immersed vertically in a liquid and has a vertex in the surface the opposite side being horizontal at a depth h Find the depth of the centre of pressure if the radius of gyration of the triangle about a horizontal line through its vertex in the plane of the triangle is $h/\sqrt{2}$

(2) Find the depth of the centre of pressure of a triangle ABC immersed vertically in a liquid with A at a depth p and BC horizontal at depth $p + h$ given that the radius of gyration of the triangle about a line through the centre of gravity parallel to BC is $h/\sqrt{18}$

(3) If in the last question BC is at a depth p and A at a depth $p + h$ find the depth of the centre of pressure

(4) ABC is a triangular area whose base is the surface of water and vertex C vertically below AB If h is the length of the perpendicular from C to AB and the area is sunk so that AB is at a depth of nh prove that the vertical displacement of the centre of pressure in the area, is $nh/2(3n + 1)$ using the result of Question 3 above

(5) One end of a trough of rectangular cross section 3 ft wide by 4 ft deep is hinged along its lower edge and is kept in position by a horizontal force P lb wt applied to the mid point of its upper edge If the trough is just filled with water find the least value of P necessary to prevent water flowing out (The density of water = 62 5 lb per cub ft) (H.S.C I)

(6) Using the result of Example 3 § 38 prove that the centre of pressure of a triangle wholly immersed vertically in a liquid is the centre of gravity of three weights placed at the middle points of the sides and proportional to the depths of those points

(7) The door of a watertight compartment in a ship is 6 ft. high and 2 ft. 6 in. wide and is hinged along a vertical edge. The top of the door is 7 ft. below the surface of the sea outside; find (i) the total thrust on the door, (ii) the reaction at the hinge.

(8) A lock-gate operates by sliding vertically upwards and is supported at its upper and lower horizontal edges. Show that, if a be the height of the lock-gate, h the depth of the water on the deeper side, h' the depth on the shallower side, l the breadth of the gate, and w the weight of unit volume of water,

the reaction on the top support $= wl\,(h^3 - h'^3)/6a$,

the reaction on the bottom support $= \frac{1}{2}wl\,(h^2 - h'^2) - wl\,(h^3 - h'^3)/6a$.

(9) A plane area is immersed in a liquid with its centre of gravity at a depth a and its centre of pressure at a depth b. If an additional layer of thickness h is now superimposed, show that the position of the centre of pressure rises a vertical distance of $h\,(b - a)/(a + h)$ relative to the area.

(10) Show that if a rectangle be immersed at different depths, always at an inclination θ to the horizontal and always with the same two edges horizontal, the centres of pressure and of gravity tend to coincide as the depth increases.

(11) A circular lamina of radius 3 in. is immersed vertically in a liquid of density ρ lb. per cub. in., its centre being at a depth 4 in. Find graphically the depth of the centre of pressure and compare the result with that obtained from Example 2, § 42.

(12) A triangle is immersed vertically in water with its base in the surface and the opposite vertex at a depth p. If the height of the effective surface for water be h, prove that the centre of pressure will be a distance $hp/(6h + 2p)$ higher than if the atmospheric pressure were neglected.

(13) A parallelogram is completely immersed vertically in a liquid with its centre at a depth h below the surface. If a and b are the lengths of the projections of its sides on a vertical line, show that the depth of the centre of pressure exceeds the depth of the centre of gravity by an amount $(a^2 + b^2)/12h$.

(14) A right-angled triangle is immersed vertically in a liquid as shown in the

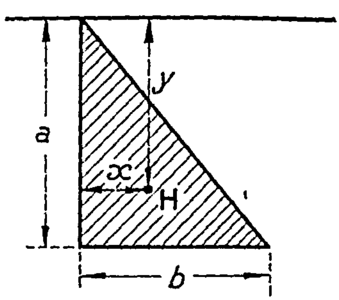


diagram. Find the position of the centre of pressure (H) by determining the distances x and y by (i) inspection, (ii) integration.

(15) A lamina in the form of a quadrant of a circle of radius r is immersed vertically in a liquid with one straight edge in the surface Find the distances of the centre of pressure from the two straight edges

(16) Prove that if a lock is closed by two equal gates meeting at an angle α, the thrust between the gates is equal to the thrust on each hinge post, and is $\frac{1}{2}$ cosec α of the hydrostatic thrust which would be exerted on a single gate, closing the lock at right angles to its length

(17) The gates of a canal lock are each $12\frac{1}{2}$ ft. wide, the width of the canal is 24 ft, the depth of the water at one side is 18, and at the other 12 ft Find the magnitude of the resultant water pressure on either gate, and show that it acts 7·6 ft from the bottom Show that the thrust between the gates themselves is about 56 ton weight

(18) A square of side a is immersed vertically in a liquid with one corner in the surface and a side inclined at an angle θ to the vertical Show that the difference in depth between the centre of gravity and centre of pressure is $a/6\,(\sin\theta + \cos\theta)$

(19) A masonry retaining wall of rectangular section is to be built 12 ft high to keep back water which may rise to the top of the wall. If the masonry weighs 150 lb per cub. ft find the minimum width of wall required in order that it shall never slide at its base, the coefficient of friction there being 0·4

Prove that, with this width, the wall will be safe against overturning

(20) If, in the last question, as a safety precaution against sliding the wall is constructed 8 ft wide, find where the resultant of the weight of the wall and the hydrostatic thrust of the water cuts the base Is any *tensile* stress produced in the wall?

(21) A drain outlet consists of a V-shaped trough with its axis horizontal and an equilateral triangular cross-section of side $2a$ It has an oblique plane end filled by a close-fitting triangular trapdoor, kept shut by its own weight. The trapdoor is hinged about its upper edge of length $2a$, which is horizontal, the remaining sides of the door each being of length $4a$ If the trapdoor opens when the depth of liquid in the trough is a, show that the weight of the trapdoor is

$$\tfrac{5}{12} w a^3 (2\sqrt{3} - 1)$$

where w is the weight of unit volume of the liquid (H.S.C., III)

(22) A vessel contains a homogeneous fluid of density ρ_1 to a depth greater than a and above this a layer of depth $d\ (> a)$ of homogeneous fluid density ρ_2 A plane circular disc of radius a is placed in the vessel with its plane vertical and its centre in the surface common to the two fluids. Ignoring atmospheric pressure, show that the depth of the centre of pressure of the disc below its centre is

$$\frac{3\pi(\rho_1 + \rho_2)a^2}{8\{3\pi\rho_2 d + 2(\rho_1 - \rho_2)a\}}$$

(H.S.C., III)

ANSWERS

1. $\dfrac{3h}{4}$. 2. $\dfrac{6p^2 + 8ph + 3h^2}{2(3p + 2h)}$. 3. $\dfrac{6p^2 + 4ph + h^2}{2(3p + h)}$.

5. 500 lb. wt. 7. (i) 9375 lb. wt.; (ii) $4687\frac{1}{2}$ lb. wt.

11. $4\frac{9}{16}$ in. 14. $x = 3b/8$, $y = 3a/4$.

15. Distance below horizontal edge $= 3\pi r/16$; distance from vertical edge $= 3r/8$.

17. 30·22 ton wt. 19. 6·25 ft.

20. 1·25 ft. from the centre of the wall, away from the water face. No, because the resultant lies within the "middle third" of the wall.

CHAPTER VII

PRESSURE OF HEAVY FLUIDS (5)

Resultant Thrust on any Surface

48 Nature of the problem

In the last three chapters we have been concerned with the thrust exerted by a fluid on a *plane* area and we now want to extend our methods to find the resultant thrust on any *curved* surface. This is more difficult, because in the case of a plane area the thrusts at various points were all parallel to each other since they were all at right angles to the area and could, therefore, be compounded into a single resultant. In the case of a curved surface, however, the thrusts are still all normal to the surface (see § 4), and consequently are in different planes so that they cannot be added together to give a single resultant. We can, however, *resolve the pressure on each* element of the surface into three components at right angles and then find the resultant component in each direction.

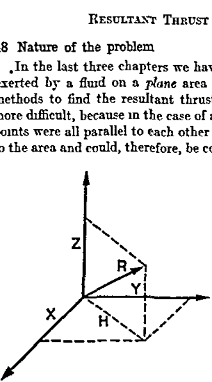


Fig 84

In general, this is as far as we are able to go and the resultant thrust on the surface will be represented by these three components in their respective planes. For instance the horizontal plane which contains one of the horizontal components will, in general, be a different plane from that which contains the other horizontal component so that their lines of action cannot intersect.

If it happens, however, that the three components *do* meet in a point (which is often the case if the body is symmetrical) then we may compound them into a single resultant thrust as follows:—

If X and Y represent the resolved components in two perpendicular horizontal directions and Z the component in a vertical direction (Fig 84) then H, the resultant of X and Y, is equal to $\sqrt{(X^2 + Y^2)}$ by the parallelogram of forces and similarly the resultant thrust (R) on the surface is the resultant of Z and H and is therefore given by

$$R = \sqrt{(X^2 + Y^2 + Z^2)}$$

We shall show, in the following paragraphs, how to find the resultant components in the vertical and horizontal directions.

49. Resultant vertical thrust

To find the resultant vertical component of the thrust of a fluid on an immersed surface (generally termed the resultant vertical thrust) we need only consider the equilibrium of a vertical column of fluid standing on the curved surface.

Let ABCD (Fig. 85) be an area on a curved surface with fluid pressing on it from above and let vertical lines be drawn through *A, B, C, D to meet the surface of the fluid in a, b, c, d.*

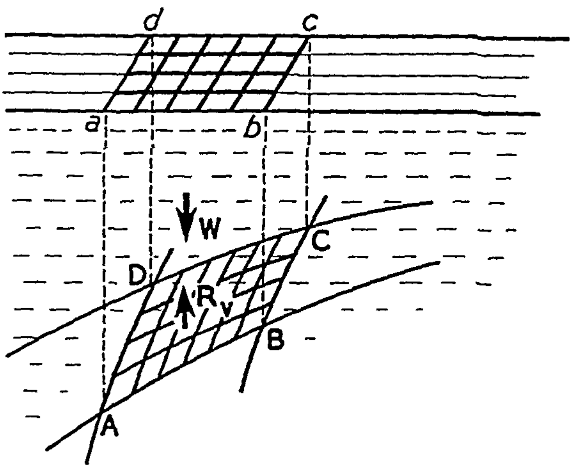


Fig. 85.

Consider the equilibrium of the fluid in the column ABCD*abcd*. The only *vertical* forces on it are:—

(1) Its weight (W), acting downwards through its centre of gravity, and

(2) The upward component of the reaction (R_v) exerted by the surface on the fluid.

Resolving vertically these are equal and opposite so that

$$R_v = W.$$

But the reaction of the surface on the fluid is equal and opposite to the thrust of the fluid on the surface. Hence *resultant vertical thrust of the fluid on the surface is equal to the weight of the column of fluid and acts downwards through the C.G. of the fluid.*

If, on the other hand, the fluid presses the surface upwards as in the case of the area chosen in Fig 86, exactly the same procedure is adopted as before The pressure intensity due to the fluid depends only on the depth below the surface and consequently the thrust on the area in this case is equal and opposite to the thrust when the fluid presses on the surface from above, i e in Fig 86 the vertical component of the thrust on the area PQRS is equal to the weight of the fluid in the column which would stand on this area up to the surface of the fluid, but acts *upwards* through the centre of gravity of this supposed column of fluid

We have already had an example of this vertical component of thrust on a curved surface in § 25, Example 1 There we had a hollow cone filled to half its height with sea water and we determined the weight of the liquid and also the thrust on the base The difference was 2·618 lb wt which we attributed to the vertical component of the thrust on the curved surface

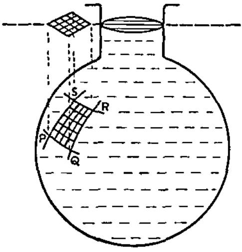


Fig 86

In accordance with the theory in this chapter we may now show that this weight is, in fact, the weight of the liquid which would stand on the curved surface Using the dimensions given in Example 1, § 25, we have —

volume of cylinder of height 6 in, standing on base of radius 3 in

$$= \pi \left(\tfrac{1}{4}\right)^2 \tfrac{1}{2} \text{ cub ft}$$

But volume of frustum of cone was $\frac{1}{3}\pi \frac{7}{128}$ cub ft,

volume of liquid which would stand on curved surface

$$= \frac{\pi}{32} - \frac{7\pi}{3 \cdot 128} \text{ cub ft}$$

$$= \frac{5\pi}{3 \cdot 128} \text{ cub ft},$$

weight of sea-water which would stand on curved surface

$$= \frac{5\pi}{3 \cdot 128} \times 64 \text{ lb wt}$$

$$= \frac{5\pi}{6} \text{ lb. wt.}$$

$$= 2{\cdot}618 \text{ lb. wt.}$$

Thus, the vertical component of the thrust exerted by the sea-water on the curved surface = 2·618 lb. wt. and acts upwards.

We leave the solution of further examples involving the vertical component of the thrust on a curved surface until we have considered the horizontal component.

50. Resultant horizontal thrust

We may determine the horizontal component of the thrust on a curved surface (the resultant horizontal thrust) by considering the

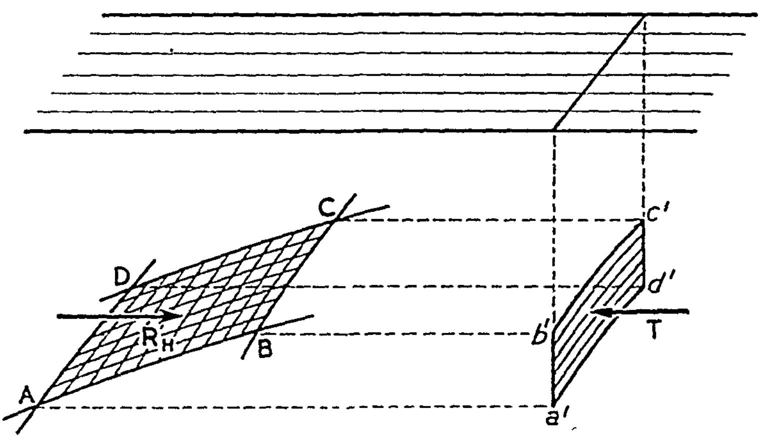


Fig. 87.

equilibrium of a horizontal column of fluid bounded at one end by the curved surface and at the other by a plane area.

Consider Fig. 87. It shows an area ABCD on a curved surface (as in § 49), but this time it is projected on to a vertical plane, the projection being $a'b'c'd'$. If we consider ABCD as a lamina completely immersed in a fluid, then we may consider the equilibrium of the column ABCD$a'b'c'd'$ of fluid. The only two forces on this column perpendicular to the plane of projection (*i.e.* along the axis of the column) are:—

(1) The thrust (T) exerted by the remainder of the fluid across the plane face $a'b'c'd'$ acting at the centre of pressure of this plane area, and

(2) The horizontal component of the reaction (R_H) exerted by the curved lamina ABCD

Hence these are equal and opposite, and since the thrust exerted by the fluid on the lamina is equal and opposite to the reaction exerted by the lamina on the fluid we have —

Resultant horizontal thrust of the fluid on the area in a given horizontal direction is equal to the thrust which would be exerted on the projection of the area on to a vertical plane perpendicular to the given direction and acts through the centre of pressure of that projected area

Example—*A hollow hemispherical shell of radius r is immersed in liquid of density ρ with a diameter vertical If the centre is at a depth h, find the resultant horizontal thrust on the hemisphere*

The projection of the hemispherical shell is a circle, radius r and centre at depth h Hence, by the theorem above, the resultant horizontal thrust on either side of the curved surface of the hemispherical shell is equal to the thrust on the circle This, from § 27 (Case 7), is $\pi r^2 \rho h$ and the depth of the centre of pressure of the circle is $h + \frac{r^2}{4h}$ from § 42 (Example 2) Thus we know the magnitude and position of the resultant horizontal thrust

51 Examples

We now propose to solve some problems concerning thrusts on curved surfaces in which we may compound the resultant vertical and horizontal thrusts to obtain a single resultant

Example 1—*A closed cylinder of radius r ft and length l ft is just full of liquid of density ρ lb per cub ft If the cylinder is held with its axis horizontal, find the liquid thrust on each half of the curved surface determined by a horizontal plane through the axis*

The resultant vertical thrust on the upper half of the curved surface of the cylinder will be the weight of the liquid which would stand on this part if the liquid were outside instead of inside, the levels in the two cases being the same Fig 88 shows the section

Thus the volume of liquid standing on the upper half

$$= 2r^2l - \tfrac{1}{2}\pi r^2 l \text{ cub ft}$$

$$= lr^2(2 - \tfrac{1}{2}\pi) \text{ cub ft},$$

$$\text{weight of liquid} = lr^2\rho(2 - \tfrac{1}{2}\pi) \text{ lb wt}$$

Hence the resultant vertical thrust on the upper part is $lr^2\rho(2 - \frac{1}{2}\pi)$ and acts upwards as shown in Fig 88 (since it acts through the centre

acts *downwards* through the centre of gravity of this column Also the resultant vertical thrust on the *lower* part of the body is equal to the weight of the column which would stand on the lower part of the surface of the body and acts *upwards* through the centre of gravity of this column of liquid,

i e the resultant vertical thrust on the lower part

= weight of column of fluid standing on body + weight of fluid displaced by body

Therefore, considering both the vertical thrusts on the upper and lower parts of the body, we have the resultant vertical thrust on the *whole* body is equal to the weight of fluid displaced by the body and acts *upwards* through the centre of gravity of this displaced fluid

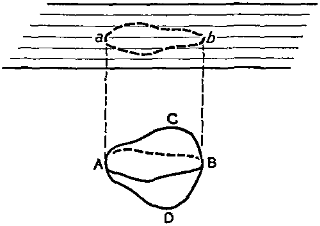


Fig 92

This point is called the *centre of buoyancy* and the resultant force vertically upwards is called the *force of buoyancy*

Since we are dealing here with a body (as distinct from a surface) there is no resultant horizontal thrust, because we may always draw a line round the surface of the body which divides it into two parts which have the same projected areas on a vertical plane so that the horizontal thrusts on each part are equal and opposite

Thus the resultant vertical thrust we obtained above is in fact the total resultant thrust of the fluid on the body

As we have already said, this principle is of very great practical importance It was first established by Archimedes and is, in consequence, known as *Archimedes' Principle*

We propose to give an alternative proof

The resultant thrust of a fluid on a solid depends on the shape and position of the solid, not on the composition of the solid itself If

we imagine the solid removed and exactly the same shape of fluid in its place then the resultant thrust is unaltered. But we now have a continuous fluid and the fluid introduced in place of the solid is in equilibrium with the rest of the fluid. This fluid has its weight acting downwards through its centre of gravity and, since it is in equilibrium with the remainder of the fluid, the resultant thrust exerted by the original fluid on the solid is equal and opposite to the weight of the fluid filling the space occupied by the solid.

The principle may be proved in exactly the same way if the solid is only partly immersed in the fluid and so, finally, we may enunciate Archimedes' Principle as follows:—

When a solid is partly or wholly immersed in a fluid at rest, the resultant thrust of the fluid on the solid is equal and opposite to the weight of the fluid displaced by the solid and acts vertically upwards through the centre of gravity of the displaced fluid.

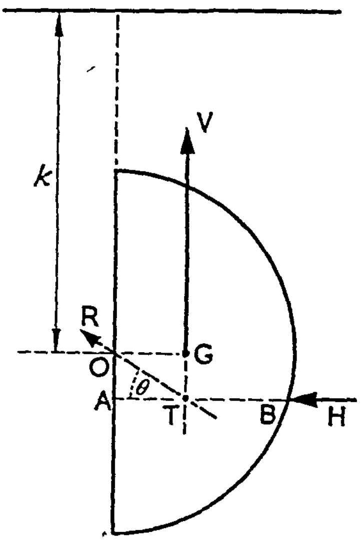


Fig. 93.

Example 1.—*A solid hemisphere of radius r is immersed in liquid of density ρ with its plane face vertical and centre at a depth k. To find the resultant thrust on the curved surface of the hemisphere.*

We have already found (§ 50, Example 1) the horizontal thrust (H) on the curved surface of a hemisphere. It is of magnitude $\pi r^2 \rho k$ and acts along the line BA (Fig. 93) where $OA = \frac{r^2}{4k}$, O being centre of the hemisphere.

From Archimedes' Principle the vertical thrust (V) on the hemisphere is equal to the weight of liquid displaced by the hemisphere (*i.e.* $\frac{2}{3}\pi r^3 \rho$) and acts upwards through the centre of gravity of this displaced liquid. If G is the centre of gravity of the hemisphere, then OG is horizontal and equals $\frac{3}{8}r$.

Since the hemisphere is symmetrical about a vertical plane dividing it into two quadrants, there is no other horizontal component and the resultant thrust (R) on the hemisphere is given by

$$\begin{aligned} R &= \sqrt{(H^2 + V^2)} \\ &= \sqrt{(\pi^2 r^4 \rho^2 k^2 + \tfrac{4}{9}\pi^2 r^6 \rho^2)} \\ &= \pi r^2 \rho \sqrt{(k^2 + \tfrac{4}{9} r^2)}. \end{aligned}$$

If the lines of action of V and H intersect in T and θ is the angle that R makes with AT, then

$$\tan\theta = \frac{V}{H} = \frac{\frac{2}{3}\pi r^3 \rho}{\pi r^2 \rho k} = \frac{2r}{3k}$$

But
$$\frac{OA}{AT} = \frac{r^2}{4k}\frac{8}{3r} = \frac{2r}{3k},$$

R passes through the centre O

This could have been deduced independently Since every thrust on the curved surface is normal to the surface, the lines of action of every thrust on every element must pass through the centre, therefore the resultant must pass through the centre

53. Resultant thrust on a curved surface enclosed by a plane curve

If we require to find the resultant thrust on a curved surface bounded by a plane curve we need not find the horizontal and vertical thrusts separately because the system is in equilibrium under the action of three forces

(1) The weight of fluid enclosed by the curved surface and plane face,

(2) The thrust across the plane face and

(3) The thrust across the curved surface

If we know the magnitudes and directions of the first two of these, we may find the third from the triangle of forces

Example —*A trough is made by cutting a right circular cone by a plane through its axis If the trough is full of liquid of density ρ, to find the resultant thrust on the curved surface*

Let Fig 94 represent the forces on the liquid, these are —

(1) The force (P) exerted by the plane semicircular face,

(2) The weight (W) of the liquid,

(3) The resultant reaction (R) of the curved surface

If the semicircular base is of radius r, then

$$P = \tfrac{1}{2}\pi r^2 . \rho \frac{4r}{3\pi} \quad \text{(Case 8, § 27)}$$

$$= \frac{2\rho r^3}{3}.$$

The weight (W) of liquid in the trough is given by $W = \frac{1}{2}.\frac{1}{3}\pi r^2 h\rho$, where h is the length of the trough.

Consequently, since W and P are at right angles, from the triangle of forces

$$R = \sqrt{(P^2 + W^2)}$$

$$= \sqrt{(\tfrac{4}{9}\rho^2 r^6 + \tfrac{1}{36}\pi^2 r^4 h^2 \rho^2)}$$

$$= \tfrac{1}{6}\rho r^2 \sqrt{(16r^2 + \pi^2 h^2)},$$

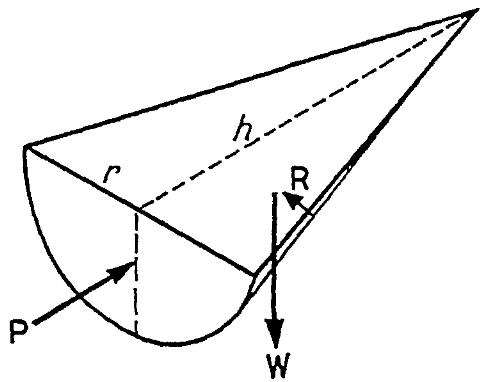


Fig. 94.

and the thrust exerted by the liquid on the curved surface is equal and opposite to the reaction exerted by the curved surface on the liquid.

The direction of the forces on the trough are shown in Fig. 95.

It is clear that we should have obtained the same result by our previous method of finding the resultant vertical and horizontal thrusts on the curved surface (V and H respectively), for

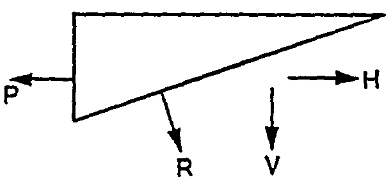


Fig. 95.

V = weight of fluid standing on curved surface, *i.e.*

$$V = W,$$

and H = thrust on projection of surface on vertical plane, *i.e.*

$$H = P,$$

and since $R = \sqrt{(V^2 + H^2)}$,

we obtain the same result as before.

To find the *direction* of R, if it makes an angle θ with the horizontal, then

$$\tan\theta = \frac{W}{P}\left(\text{or } \frac{V}{H}\right)$$

$$= \frac{\frac{1}{6}\pi r^2 h\rho}{\frac{2}{3}\rho r^3}$$

$$= \frac{\pi h}{4r}.$$

We may also find the *line of action* of the resultant thrust on the curved surface. The centre of gravity of the liquid in the half cone is also $\frac{1}{4}h$ from the base and the centre of pressure of the semicircle is at a distance $\frac{3\pi r}{16}$ from the centre (see Example 2, § 47). Thus, if in Fig 96, B is the centre of pressure of the semicircle, C is the point of intersection of P and W, and A is the point in the plane of the semicircle where the line of action of R cuts it, then ABC is a triangle of force and we have

$$\frac{W}{AB}=\frac{P}{BC},$$

$$\text{or}\quad AB=\frac{W}{P}\times BC$$

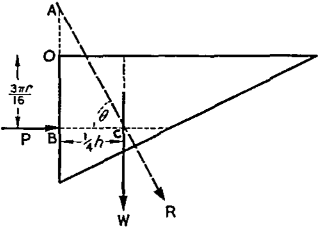


Fig 96

$$=\frac{W}{P}\times\tfrac{1}{4}h$$

$$=\frac{\pi h}{4r}\times\tfrac{1}{4}h$$

$$=\frac{\pi h^2}{16r}$$

Consequently the distance of A from the centre of the semicircle is

$$\frac{3\pi r}{16}\sim\frac{\pi h^2}{16r}$$

$$=\frac{\pi}{16r}(3r^2\sim h^2),$$

showing that the line of action of the resultant thrust on the curved

surface passes through O, the centre of the semicircle, when

$$3r^2 = h^2,$$

i.e. when $$r/h = 1/\sqrt{3},$$

i.e. when the semi-vertical angle of the cone is 30°.

Exercises VII

(1) A tank is made in the shape of a half-cylinder of length 6 ft. with plane semicircular ends of radii 2 ft. If it stands on a semicircular end and is just full of liquid of density 70 lb. per cub. ft., find the resultant horizontal thrust on the curved surface.

(2) A thin, hollow hemispherical container rests with its circular base of radius r ft. on a horizontal plane and is just full of water of density ρ lb. per cub. ft. Find in magnitude, direction and position the following forces:—

(i) the resultant vertical component of the thrust on the curved surface;
(ii) ,, ,, horizontal ,, ,, ,, ,, ,, ,, ,, ,,
(iii) the resultant thrust on the curved surface;
(iv) ,, ,, ,, ,, ,, plane base.

(3) An open-topped tank is made in the form of half a cone by bisecting a right cone by a plane through its axis. It is kept with its triangular face vertical and vertex downwards and is full of liquid of density ρ. If the radius of the semicircular top is r, and the height is h, find the magnitude of

(i) the resultant vertical thrust on the curved surface:
(ii) ,, ,, horizontal ,, ,, ,, ,, ,,
(iii) the resultant thrust on the curved surface;
(iv) the angle θ this resultant makes with the horizontal.

(4) A hemispherical bowl holding 4 lb. of a liquid is held with its rim against a vertical wall. Find the magnitude of the resultant thrust of the liquid (i) on the wall; (ii) on the bowl.

(5) A cylindrical boiler has hemispherical ends and is mounted with its axis horizontal, the overall length being twice its maximum cross-sectional diameter. When it is just full of water, show that the horizontal component of the thrust on a hemispherical end is 0·3W, where W is the weight of water contained in the boiler.

(6) A cone, whose height is 3 cm. and the area of whose base is 10 sq. cm., is filled with water and placed vertex upwards on a horizontal table. Find the resultant thrust (i) on the base; (ii) on the curved surface.

(7) A hollow cone, whose height is 4 in. and the radius of whose base is 3 in., is fixed with its base horizontal and its vertex downwards. The cone is filled with water; find the resultant thrust on the curved surface.

(8) If the cone in the last question be inverted so as to stand on its base, find the increase of the resultant thrust on the curved surface.

(9) An open horizontal channel of length a is full of liquid of density ρ. The cross-section is a quadrant of a circle of radius r, and the plane side of the channel is vertical. Find the magnitude and direction of the resultant thrust of the liquid on the curved surface, neglecting atmospheric pressure.

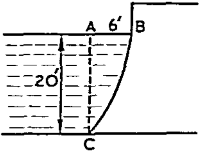
(10) A solid hemisphere whose radius is 7 cm. is immersed in liquid of specific gravity 1·5, with its curved surface uppermost and its plane surface horizontal at a depth of 20 cm. Find the resultant vertical thrusts on both the plane and the curved surfaces.

(11) A hollow cone of inner radius 4 ft. and inner height 10 ft., and not closed by a base, is placed with its rim on a horizontal plane; the cone is filled with water through a small hole at the vertex and the water does not flow out. Find the force, in tons weight, with which the water tends to lift the cone.

(12) A closed cylinder of radius r and length l is just full of liquid weighing w per unit volume. If the cylinder is held with its axis horizontal, find the liquid thrust on the lower half of the curved surface determined by the horizontal plane through the axis.

If the cylinder be tilted until the plane ends are inclined at 60° to the horizontal, find the liquid thrust on the lower plane end. (Inter. Sc.)

(13) The diagram shows part of the section of an embankment wall with a curved side BC. If AC is vertical and the area of the section ABC



is given as 80 sq. ft., find, from the dimensions indicated, the magnitude of the resultant vertical and horizontal components of the thrust on the embankment per foot length. Hence find the magnitude and direction of the resultant thrust per unit length of wall.

(14) A sphere of 1 ft. radius is immersed in water with its highest point in the surface. Taking the density of water as 62·5 lb. per cub. ft., find the resultant thrust of the water (i) on the upper (ii) on the lower half of the surface of the sphere. (H.S.C., I.)

(15) A bowl in the shape of a hemisphere is filled with water. Find the vertical thrust and the horizontal thrust on either of the portions into which it is divided by a vertical plane through its centre, in terms of W, the weight of water in the hemisphere.

[The centre of gravity of a semicircle of radius r is at a distance $4r/(3\pi)$ from the centre.]

(16) A solid right circular cone of height 1 ft. and vertical angle 60° is immersed in water with its axis vertical and its vertex in the surface. Find the direction and magnitude of the resultant thrust on one portion of its curved surface which is cut off by a plane through the axis. (H.S.C., I.)

(17) A closed vessel is in the form of a right circular cylinder with one end plane and the other a hemisphere. If the vessel rests just full of liquid with its axis horizontal, show that the ratio of the thrusts of the liquid on the hemispherical and plane ends is approximately 6 : 5.

If the ratio of the resultant thrust of the liquid on the whole surface of the vessel to the thrust on the plane end is 17 : 3, find the ratio of the total length of the vessel to its radius. (H.S.C., I.)

(18) A closed circular cylinder is full of water and hangs freely from a point in its upper rim. If the radius of its cross-section is half its length, prove that the vertical and horizontal components of the resultant thrust on its curved surface are each half the weight of the water it contains.

(19) A spherical shell is made of two equal hemispheres in contact along a vertical plane and hinged at the highest point of their rims. If the shell is full of water and is suspended from the hinge, show that the two hemispheres will not separate if $W' > 3W$, where W' is the weight of the whole shell and W the weight of water it contains.

Show also that the resultant liquid thrust on either hemisphere is $\frac{W}{4}\sqrt{13}$.

(20) A solid hemisphere of radius a is immersed in a liquid, the depth of the centre of the plane surface being h below the free surface of the liquid and the plane of the base being inclined at an angle θ to the horizontal. By considering the liquid pressures acting on the hemisphere, show that the centre of pressure of the circular base is $a^2\sin\theta/4h$ from the geometrical centre. (H.S.C., III.)

(21) A cylindrical vessel full of water is held with its axis inclined at an angle of 45° to the vertical. Find the magnitudes of the pressures on the ends, and show that the resultant pressure on the curved surface will equal the difference between the pressures on the ends.

(22) A uniform solid hemisphere, of weight W, floats with its curved surface partly immersed in a liquid and with one point of its rim in the surface, equilibrium being maintained by a vertical force pW applied at this point. Find the value of p if the plane face of the hemisphere is inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$.

Find also the ratio of the density of the material of the hemisphere to that of the liquid.

[Assume, without proof, that the volume of a cap of height h of a sphere of radius a is $\pi h^2(a - h/3)$ and that for a hemisphere of radius a, the distance of the C.G. from the centre of the plane face is $3a/8$.]

(H.S.C., I.)

(23) A hemisphere, of radius r, is immersed in water with its plane base inclined at an angle θ to the horizontal and its centre at a depth h ($> r$), the curved surface being uppermost. Show that the resultant thrust on the

curved surface is a force acting through the centre at an inclination to the horizontal of

$$\tan^{-1}\left[\left(\frac{2r}{3h}\right)\operatorname{cosec}\theta - \cot\theta\right]$$

(24) A solid sphere is divided into eight equal parts by three mutually perpendicular planes passing through the centre One of these parts is immersed in water with a plane face in the surface Prove that the resultant thrust on the curved surface is $W(\pi^2+8)^{\frac{1}{2}}/\pi$ where W is the weight of water displaced by the part

(25) If the part of the sphere considered in the last question is lowered without turning through a distance h, show that the thrusts on the three plane faces reduce to a single force, and find its magnitude

(26) The bottom of a glass is a circle of an inch diameter, the side forming a portion of a right circular cone of semi vertical angle 30°, with vertex downwards The glass is filled to a height of 6 in with water Find approximately in ounces the resultant pressure on the side of the glass having given that a cubic foot of water weighs 1000 ounces

(27) A solid cone of vertical angle 2α is immersed in a liquid with a generating line in the surface Show that the resultant thrust on the curved surface makes an angle θ with the horizontal given by

$$\tan\theta = \tfrac{4}{3}\operatorname{cosec}2\alpha - \tan\alpha$$

If this resultant thrust is horizontal show that its magnitude is equal to the weight of a hemisphere of the liquid with radius equal to that of the base of the cone

ANSWERS

1 5040 lb wt

2 (i) $\pi r^3\rho/3$, vertically upwards through centre of sphere (ii) 0 (iii) same as (i) (iv) $\pi r^3\rho$ vertically downwards through centre of sphere

3 (i) $\pi r^2 h\rho/6$ downwards (ii) $rh^2\rho/3$ horizontally

(iii) $rh\rho(4h^2+\pi^2r^2)^{\frac{1}{2}}/6$ (iv) $\tan^{-1}\left(\frac{\pi r}{2h}\right)$

4 (i) 6 lb wt (ii) 7 21 lb wt

6 (i) 30 grm wt (ii) 20 grm wt acting vertically upwards

7 21 8 oz wt 8 21 8 oz wt

9 0 93 $r^2a\rho$, 57° 30 with the horizontal

10 4 62 kilog, 3 54 kilog 11 9 35 ton wt

12 $r^2lw(\pi+4)/2$, $\pi r^2w(l+r\sqrt{3})/2$

13 Horizontal component = 12 500 lb wt vertical component = 5000 lb wt Resultant thrust = 13 463 lb wt downwards at 21° 48 to the horizontal

14 (i) 65 45 lb wt, (ii) 327 25 lb wt

15 Vertical thrust = W/2 horizontal thrust = W/π

16 32 476 lb wt at 47° 48 with downward vertical 17 6 1

21 $\pi r^2\rho/\sqrt{2}$, $\pi r^2\rho(r+l)/\sqrt{2}$ where r = radius of circular cross section l = length of cylinder, ρ = density of water

22 $p = 9/32$, 0 02 1

25 $\rho r^2(27\pi^2h^2+48\pi rh+32r^2)/12$ where r = radius of the sphere

26 62 5 oz wt

CHAPTER VIII

EQUILIBRIUM OF FLOATING BODIES

54. Conditions of equilibrium of a body floating freely in a liquid

The last chapter was concerned with the resultant thrust of a fluid on any surface and in § 52 we considered the case of a closed surface, *i.e.* an immersed body. We deduced Archimedes' Principle, which, because of its importance, we quote again:—When a solid is partly or wholly immersed in a fluid at rest, the resultant thrust of the fluid on the solid is equal and opposite to the weight of the fluid displaced by the solid and acts vertically upwards through the centre of gravity of the displaced fluid.

Since this was proved true for all fluids, it is equally true for gases as for liquids. In this section we deal with the conditions of equilibrium of a body floating freely in a liquid and leave the consideration of gases until Chapter X.

Let Fig. 97 represent the section of a body floating freely in a liquid of density ρ.

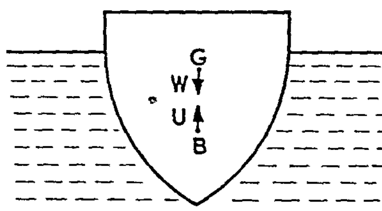


Fig. 97.

There are only two vertical forces acting on the body, these are:—

(1) The weight of the body (W) acting downwards through the centre of gravity (G) of the body; and

(2) The upthrust or force of buoyancy (U) which is equal in magnitude to the weight of the displaced liquid, and acts upwards through the centre of buoyancy (B), *i.e.* the centre of gravity of the displaced liquid.

For equilibrium these two forces must be equal and act in opposite directions in the same vertical line, hence, for equilibrium

(1) $$W = U;$$

(2) B and G are in the same vertical line.

We now give some examples illustrating this principle.

Example 1.—*A solid wooden cylinder, 4 ft. long, floats in water with its axis vertical and a depth of 3 ft. immersed. Find the specific gravity of the wood.*

All we require to do is to equate the weight of the cylinder to the weight of water displaced.

Thus if Fig 98 represents the cylinder supposing the section of the cylinder to be of radius r ft we have

$$\text{volume of cylinder} = \pi r^2 \; 4 \text{ cub ft}$$

$$\text{weight of cylinder} = \pi r^2 \; 4 \; s \; 62\tfrac{1}{2} \text{ lb wt} \qquad \text{(i)}$$

where s is the specific gravity of the wood

Also

$$\text{volume of water displaced} = \pi r^2 \; 3 \text{ cub ft}$$

$$\text{weight of displaced water} = \pi r^2 \; 3 \; 62\tfrac{1}{2} \text{ lb wt} \qquad \text{(ii)}$$

Equating (i) and (ii)

$$\pi r^2 \; 4 \; 62\tfrac{1}{2} s = \pi r^2 \; 3 \; 62\tfrac{1}{2}$$

$$4s = 3$$

i e $$s = \tfrac{3}{4}$$

Thus the specific gravity of the wood is $\frac{3}{4}$ or in other words it weighs $\frac{3}{4} \times 62\frac{1}{2}$ (i e $46\frac{7}{8}$) lb per cub ft

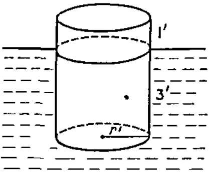


Fig 98

Example 2 —*If the specific gravity of ice is 0 918 and that of sea water 1 026 show that of the whole volume of an iceberg $\frac{2}{19}$ will be above water*

Let the iceberg have a volume of V cub ft of which v cub ft are above the surface of the sea

Then weight of iceberg $=$ V (0 918) $62\frac{1}{2}$ lb wt

and the volume of sea water displaced is $\mathrm{V} - v$

so that weight of sea water displaced $= (\mathrm{V} - v)(1\ 026)\ 62\frac{1}{2}$ lb wt

Since these weights are equal

$$(\mathrm{V} - v)(1\ 026)\ 62\tfrac{1}{2} = \mathrm{V}\,(0\ 918)\ 62\tfrac{1}{2}$$

$$\mathrm{V}\,(1\ 026 - 0\ 918) = 1\ 026v$$

or $$0\ 108\mathrm{V} = 1\ 026v$$

$$\frac{v}{\mathrm{V}} = \frac{0\ 108}{1\ 026}$$

$$= \tfrac{2}{19}$$

i e $$v = \tfrac{2}{19}\mathrm{V}$$

i e $\frac{2}{19}$ of the whole volume of an iceberg is above water

Example 3 —*What is the volume of a mine which weighs 1 ton and just floats completely immersed in sea-water weighing 64 lb per cub ft ?*

If the volume of the mine is V cub. ft., then the weight of sea-water displaced $= 64V$ lb. wt.

$$\text{But the weight of the mine} = 2240 \text{ lb. wt.};$$

$$\therefore \quad 64V = 2240;$$

$$\therefore \quad V = \frac{2240}{64} = 35 \text{ cub. ft.}$$

Example 4.—*A uniform circular cylinder, of specific gravity 0·9 floats upright in salt water (specific gravity 1·03). When transferred to fresh water the cylinder still floats upright but is immersed an extra 3 in. If the area of the cross-section is 1 sq. ft., find the volume of the cylinder.* (*Inter. Sc.*)

Let Figs. 99 and 100 represent the cylinder floating in salt water and fresh water respectively. Suppose the length of the cylinder is

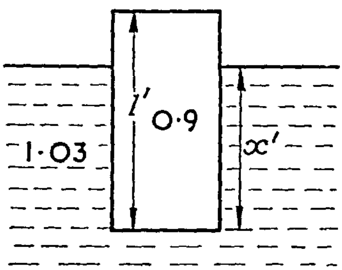


Fig. 99.

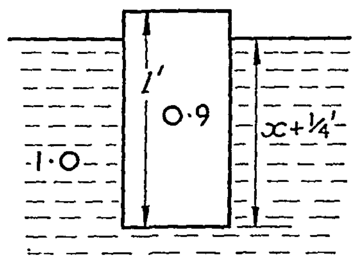


Fig. 100.

l ft. and that it has x ft. immersed when in salt water and consequently, from the information given in the question, $(x + \frac{1}{4})$ feet immersed when in fresh water.

The weight of the cylinder is, of course, independent of the liquid in which it is floating.

Thus, volume of cylinder $= l$ cub. ft.,

since the cross-section area is 1 sq. ft.;

$$\therefore \quad \text{weight of cylinder} = l\,(0{\cdot}9)\,w \text{ lb. wt.} \;\ldots\ldots\ldots \text{(i)}$$

where w is the weight of 1 cub. ft. of fresh water. (We do not even need to know that $w = 62{\cdot}5$ lb. wt.)

When the cylinder is immersed in sea-water, the liquid displaced is also a cylinder of 1 sq. ft. cross-section, but its length is x ft.

Therefore, volume of sea-water displaced $= x$ cub. ft.;

$$\therefore \quad \text{weight of sea-water displaced} = x\,(1{\cdot}03)\,w \text{ lb. wt.} \;\ldots\ldots \text{(ii)}$$

Equating (i) and (ii), since the weight of the cylinder is equal to the weight of the sea water displaced,

$$l(0.9)w = x(1.03)w,$$

or $$1.03x = 0.9l \qquad \text{(iii)}$$

This provides one equation between x and l; to obtain a second equation we must consider the weight of fresh water displaced.

Referring to Fig. 100,

volume of fresh water displaced $= (x + \frac{1}{4})$ cub. ft.,

weight of fresh water displaced $= (x + \frac{1}{4})(1.0)w$ lb. wt. (iv)

and this must also equal the weight of the cylinder, so that, equating (i) and (iv),

$$l(0.9)w = (x + \tfrac{1}{4})(1.0)w,$$

or $$x + 0.25 = 0.9l \qquad \text{(v)}$$

We now have two equations, (iii) and (v), to determine l.

From (v) $$x = 0.9l - 0.25$$

Substituting for x in (iii),

$$1.03(0.9l - 0.25) = 0.9l,$$

$$0.9 \times 0.03l = 1.03 \times 0.25,$$

$$l = 9.54 \text{ ft.}$$

Therefore the volume of the cylinder is 9.54 cub. ft.

Example 5.—*A hollow sphere of external diameter 2a in. is made of material of thickness t in. and of specific gravity 4. The sphere floats half immersed in liquid of specific gravity 1.141. Prove that t = 0.05a.* (*Inter. Sc.*)

The liquid displaced is a hemisphere of radius a; therefore

volume of liquid displaced $= \frac{2}{3}\pi a^3$ cub. in.,

∴ weight of liquid displaced $= \frac{2}{3}\pi a^3(1.141)w$ lb. wt. (i)

where w is the weight of 1 cub. in. of water.

The outer radius of the spherical shell is a and the inner radius is $a - t$, so that

volume of spherical shell $= \frac{4}{3}\pi a^3 - \frac{4}{3}\pi(a - t)^3$ cub. in.

$= \frac{4}{3}\pi(3a^2t - 3at^2 + t^3)$ cub. in.,

∴ weight of spherical shell $= \frac{4}{3}\pi(3a^2t - 3at^2 + t^3)4w$ lb. wt. (ii)

Equating (i) and (ii),

$$\tfrac{2}{3}\pi a^3(1.141)w = \tfrac{4}{3}\pi(3a^2t - 3at^2 + t^3)4w,$$

or $$a^3(1.141) = 8(3a^2t - 3at^2 + t^3) \qquad \text{(iii)}$$

This is a cubic equation in t, but since we only need to check that $t = 0{\cdot}05a$, we may substitute this value of t in the right-hand side of (iii) and confirm that its value is $1{\cdot}141a^3$.

Then right-hand side of (iii) becomes $8\left(3a^2 \,.\, \dfrac{a}{20} - 3a \,.\, \dfrac{a^2}{400} + \dfrac{a^3}{8000}\right)$

$$= a^3\,(1{\cdot}2 - 0{\cdot}06 + 0{\cdot}001)$$
$$= 1{\cdot}141a^3.$$

Thus, $t = \frac{1}{20}\,a$ is a solution of (iii).

It should be noticed, however, that we may avoid the awkward cubic equation in this case by first of all writing the inner radius of the shell as x. Then

$$\text{volume of spherical shell} = \tfrac{4}{3}\pi a^3 - \tfrac{4}{3}\pi x^3,$$

so that

$$\text{weight of spherical shell} = \tfrac{4}{3}\pi\,(a^3 - x^3)\,.\,4w \text{ lb. wt.},$$

in place of (ii) above, and then equating this to (i),

$$\tfrac{2}{3}\pi a^3\,(1{\cdot}141)\,w = \tfrac{16}{3}\pi\,(a^3 - x^3)\,w,$$

i.e.

$$1{\cdot}141a^3 = 8a^3 - 8x^3;$$
$$\therefore\quad 8x^3 = 6{\cdot}859a^3;$$
$$\therefore\quad 2x = 1{\cdot}9a;$$
$$\therefore\quad x = 0{\cdot}95a,$$

and

$$t = a - x = 0{\cdot}05a.$$

Example 6.—*A light conical shell, made of material whose thickness can be neglected, has a particle of weight W attached to its vertex and floats in water with its vertex downwards and a length x of its axis, which is vertical, immersed. Liquid of specific gravity s is poured into the cone until the level of the liquid and the level of the water are the same. Show that the length y of the axis now immersed is given by*

$$s = 1 - x^3/y^3. \qquad \textit{(Inter. Sc.)}$$

Let Fig. 101 represent the empty cone floating in water and Fig. 102 when liquid is poured in so that the levels of liquid and water are the same. Suppose that the radii of the circular sections of the cone at the surface of the water are r_1 and r_2 respectively in the two cases.

Then, from Fig. 101,

$$\text{volume of water displaced} = \tfrac{1}{3}\pi r_1^2 x;$$
$$\therefore\quad \text{weight of water displaced} = \tfrac{1}{3}\pi r_1^2 x w,$$

where w is the weight per unit volume of the water.

Since the conical shell is of negligible weight, the only downward force is the weight (W) attached to the vertex, hence

$$W = \text{weight of water displaced}$$
$$= \tfrac{1}{3}\pi r_1^2 x w \qquad \text{(i)}$$

From Fig 102,

$$\text{volume of water now displaced} = \tfrac{1}{3}\pi r_2^2 y,$$
$$\text{weight of water displaced} = \tfrac{1}{3}\pi r_2^2 y w$$

The total downward force has now been increased by the weight of the liquid in the cone, *i e* by $\frac{1}{3}\pi r_2^2 ysw$, since the specific gravity of this liquid is s

Hence the total downward force now is

$$W + \tfrac{1}{3}\pi r_2^2 ysw,$$

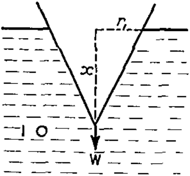


Fig 101

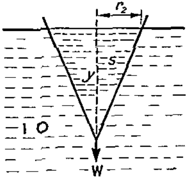


Fig 102

and this equals the new weight of water displaced

i e $$W + \tfrac{1}{3}\pi r_2^2 ysw = \tfrac{1}{3}\pi r_2^2 yw \qquad \text{(ii)}$$

Eliminating W from (i) and (ii),

$$\tfrac{1}{3}\pi r_1^2 xw + \tfrac{1}{3}\pi r_2^2 ysw = \tfrac{1}{3}\pi r_2^2 yw$$

i e $$r_1^2 x + r_2^2 ys = r_2^2 y,$$

$$s = 1 - \frac{r_1^2 x}{r_2^2 y} \qquad \text{(iii)}$$

From similar triangles $$\frac{r_1}{x} = \frac{r_2}{y},$$

$$\frac{r_1^2}{r_2^2} = \frac{x^2}{y^2}$$

and (iii) becomes $$s = 1 - x^3/y^3$$

Example 7 —*A piece of iron (specific gravity 7 8) is imbedded in a spherical lump of ice (specific gravity 0 918) of radius 10 cm which floats in water with 1/20th of its volume above the surface Find the volume of the iron*

If the ice slowly melts, its shape remaining spherical, find the radius of the sphere when it sinks. (*Inter. Sc.*)

$$\text{The volume of water displaced} = \tfrac{19}{20} \times \text{volume of sphere}$$
$$= \tfrac{19}{20} \cdot \tfrac{4}{3}\pi (10)^3 \text{ c.cm.};$$
$$\therefore \quad \text{weight of water displaced} = \tfrac{19}{20} \cdot \tfrac{4}{3}\pi (10)^3 \text{ grm. wt.},$$

since 1 c.cm. of water weighs 1 grm.

If the volume of the iron is V c.cm., then the volume of ice is $\frac{4}{3}\pi (10)^3 - V$ c.cm., so that

$$\text{weight of iron} = V (7{\cdot}8) \text{ grm. wt.},$$

and
$$\text{weight of ice} = [\tfrac{4}{3}\pi (10)^3 - V](0{\cdot}918) \text{ grm. wt.}$$

Since

weight of iron + weight of ice = weight of water displaced,

we have

$$7{\cdot}8V + 0{\cdot}918 [\tfrac{4}{3}\pi (10)^3 - V] = \tfrac{19}{20} \cdot \tfrac{4}{3}\pi (10)^3,$$

from which
$$V = 19{\cdot}5 \text{ c.cm.}$$

If the sphere sinks when its radius is r cm., then

$$\text{weight of water displaced} = \tfrac{4}{3}\pi r^3 \text{ grm. wt.},$$
$$\text{weight of iron} = 7{\cdot}8 \times 19{\cdot}5 \text{ grm. wt.},$$

and
$$\text{weight of ice} = (\tfrac{4}{3}\pi r^3 - 19{\cdot}5)(0{\cdot}918) \text{ grm. wt.}$$

We must still have

weight of iron + weight of ice = weight of water displaced,

therefore

$$7{\cdot}8 \times 19{\cdot}5 + (\tfrac{4}{3}\pi r^3 - 19{\cdot}5)(0{\cdot}918) = \tfrac{4}{3}\pi r^3,$$

i.e.
$$\tfrac{4}{3}\pi r^3 (1 - 0{\cdot}918) = 7{\cdot}8 \times 19{\cdot}5 + 0{\cdot}918 \times 19{\cdot}5,$$

giving
$$r = 7{\cdot}3 \text{ cm.}$$

Example 8.—*A hollow cylinder of external and internal radii a and b is open at the top and the thickness of its base is c. The cylinder floats in water with its axis vertical and a depth h immersed. If the cylinder develops a small leak show that it will never sink if its height is greater than $(a^2h - b^2c)/(a^2 - b^2)$.* (*H.S.C.*)

When the cylinder floats with a depth h of its axis immersed,

$$\text{volume of water displaced} = \pi a^2 h;$$
$$\therefore \quad \text{weight of water displaced} = \pi a^2 h w,$$

where w is the weight per unit volume of water.

If the material of the cylinder has a specific gravity s and its height is H, then

$$\text{volume of cylinder} = \pi a^2 H - \pi b^2 (H - c)$$
$$= \pi [H (a^2 - b^2) + b^2 c];$$
$$\therefore \quad \text{weight of cylinder} = \pi [H (a^2 - b^2) + b^2 c]\, sw.$$

Therefore, when cylinder floats,

$$\pi [H(a^2 - b^2) + b^2c] sw = \pi a^2 hw,$$

or

$$s = \frac{a^2h}{H(a^2 - b^2) + b^2c} \qquad (1)$$

When the cylinder develops a leak, it will not sink if

weight of cylinder < upthrust when cylinder is full of water,

i e when

weight of cylinder < weight of water displaced by whole of material of cylinder,

i e when

$$\pi [H(a^2 - b^2) + b^2c] sw < \pi [H(a^2 - b^2) + b^2c] w,$$

i e when $s < 1$,

and from (1) s is less than 1 if

$$H(a^2 - b^2) + b^2c > a^2h,$$

i e if

$$H > \frac{a^2h - b^2c}{a^2 - b^2}$$

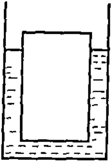


Fig 103

55 Force of buoyancy

We have already seen that the force of buoyancy is equal in magnitude to the weight of the displaced liquid," but this expression requires a little more consideration

Consider Fig 103 It represents a cylinder floating in liquid in a cylindrical vessel of only very slightly larger diameter It is clear, from the figure, that the volume of the solid displaced may easily be greater than the whole volume of liquid present Hence the "weight of displaced liquid" may be greater than the total weight of liquid present What must be understood by the term "weight of displaced liquid" is the weight of the liquid which would fill the space occupied by the immersed part of the floating body

Example —*A cylindrical vessel whose internal diameter is 6 in contains a column of water $\frac{1}{4}$ in high, a solid cylinder of wood whose diameter is $5\frac{3}{4}$ in, height 6 in, and specific gravity $\frac{1}{2}$, is lowered into the vessel, will it float or touch the bottom of the vessel?*

The volume of the cylinder of wood

$$= \pi \left(\frac{5\frac{3}{4}}{2}\right)^2 \times 6 \text{ cub in},$$

$\therefore$ weight of cylinder of wood

$$= \pi\left(\tfrac{23}{8}\right)^2 \times 6 \times \tfrac{1}{3}w = 16{\cdot}5\pi w,$$

where w = weight of a cubic inch of water.

The volume of water in the vessel

$$= \pi . 3^2 . \tfrac{1}{4} \text{ cub. in.}$$
$$= \tfrac{9}{4}\pi \text{ cub. in.}$$

If the cylinder of wood touches the bottom of the vessel, and h is the height to which the water rises, we have, since the volume of the water is unaltered,

$$\pi\left\{3^2 - \left(\frac{5\frac{3}{4}}{2}\right)^2\right\}h = \tfrac{9}{4}\pi;$$

$$\therefore\quad h = \tfrac{9}{4}/\left\{\left(\tfrac{24}{8}\right)^2 - \left(\tfrac{23}{8}\right)^2\right\}$$
$$= \tfrac{9}{4} \times \tfrac{64}{47}$$
$$= \tfrac{144}{47};$$

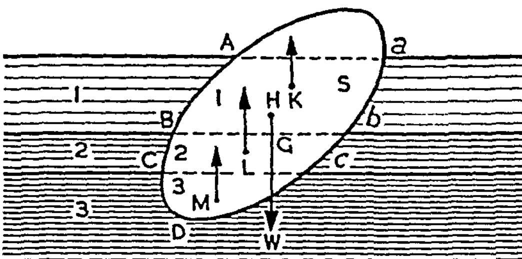


Fig. 104.

$\therefore$ weight of water that would then be displaced

$$= \pi\left(\tfrac{23}{8}\right)^2 . \tfrac{144}{47}w$$
$$= 25{\cdot}3\pi w.$$

This is greater than the weight of the cylinder of wood.

Hence the cylinder will float.

56. Conditions of equilibrium of a body floating freely in several liquids which do not mix

Let S (Fig. 104) be any solid floating partly immersed in several different liquids 1, 2, 3, . . . bounded by the horizontal planes Aa, Bb, Cc. Then it is clear, as in the foregoing investigations, that the equilibrium will be unaffected by removing the solid S and supposing the space ABba filled with the liquid 1; the space BCcb filled with the liquid 2; the space CDc filled with the liquid 3, and so on. The liquids that would fill these spaces are the *liquids displaced* by the solid, and the resultant upward thrust of the liquid on S is the resultant of the

weights of the liquids displaced acting vertically through their respective centres of gravity Hence, for equilibrium *the weight of the solid must be equal to the sum of the weights of the different liquids displaced*

The *centre of buoyancy* in this case is the centre of gravity of the whole series of liquids displaced This point and the centre of gravity of the solid must be in the same vertical line

Example 1—*Water being poured on the top of mercury (specific gravity 13 6), to find the specific gravity of a body which floats with one third of its volume above water, one third immersed in the water, and the remaining third immersed in the mercury*

Here the volumes of the water and mercury displaced are equal, and their densities are as 1 13 6

Let weight of water displaced $= w$

Then weight of mercury displaced $= 13\,6w$

But the solid is floating in equilibrium

the weight of the solid

$= $ weight of mercury displaced $+$ weight of water displaced

$= 13\,6w + w = 14\,6w$

Again one-third of the volume of the solid is immersed in water;

volume of solid $= 3$ volumes of water displaced,

weight of an equal volume of water $= 3w$

required specific gravity of solid

$$= \frac{\text{weight of solid}}{\text{weight of equal volume of water}} = \frac{14\,6w}{3w} = \frac{14\,6}{3} = 4\,8\dot{6}$$

Example 2—*A cone whose specific gravity is 2 575 rests partly immersed in water and partly in mercury To find what fraction* (i) *of its volume,* (ii) *of its axis is immersed in mercury taking the axis vertical and vertex downwards*

(i) Let V be the volume of the cone, x that of the portion submerged in mercury

Then the weights of the cone, the mercury displaced, and the water displaced, are proportional to $2\,575V$, $13\,6x$, $V - x$

For equilibrium the former equals the sum of the two latter weights,

$$\therefore \quad 2\,575V = 13\,6x + V - x,$$

$$12\,6x = 1\,575V \quad \text{or} \quad x = \cdot125V = \tfrac{1}{8}V$$

Therefore $\frac{1}{8}$ of the volume is immersed in mercury

(ii) This portion is a cone with the same vertical angle as the original cone Now it is known that *the volumes of two such cones are proportional to the cubes of their heights*

Therefore $\sqrt[3]{\frac{1}{8}}$ or $\frac{1}{2}$ of the axis is immersed in mercury

Example 3.—*If a cylinder of height h and density ρ floats just immersed, with its axis vertical, in a vessel containing two liquids, of densities σ_1 and σ_2 $(\sigma_2 > \sigma_1)$, which do not mix, find the thickness of the upper layer of liquid.*

If Fig. 105 represents the cylinder, then the liquid of density σ_1 will be the upper liquid. Suppose the thickness of this layer of liquid is x.

Then we have

weight of cylinder = weight of liquid (1) displaced + weight of liquid (2) displaced,

i.e. $$\pi r^2 h\rho = \pi r^2 x\sigma_1 + \pi r^2 (h - x)\,\sigma_2,$$

where r is the radius of the section of the cylinder;

$$\therefore \quad h\rho = x\sigma_1 + h\sigma_2 - x\sigma_2;$$

$$\therefore \quad x(\sigma_2 - \sigma_1) = h(\sigma_2 - \rho);$$

$$\therefore \quad x = \frac{h(\sigma_2 - \rho)}{\sigma_2 - \sigma_1},$$

and this is the thickness of the upper layer of liquid.

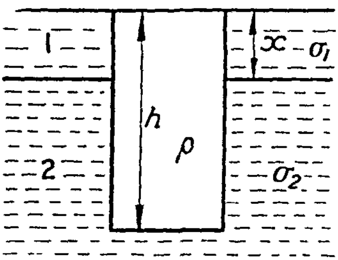


Fig. 105.

57. Correction for displaced air when body floats partly immersed in a liquid

In this chapter we have been concerned with the equilibrium of bodies floating in liquids, but the theory we have considered is true of fluids in general. Consequently § 56 is true also for a body floating partly immersed in a liquid and partly immersed in a gas.

Thus, to be strictly accurate, when a body is floating partly immersed in a liquid in an open vessel, then it also has part surrounded by (or immersed in) air, and so

weight of body = weight liquid displaced + weight air displaced.

Because the weight of the air displaced is generally very small compared with the other weights involved, it is frequently omitted (as in the examples in § 54), but it is important to realise that there is a small inaccuracy involved in this. In order to see what difference this correction makes in a particular case, we propose to re-work Example 1 of § 54, allowing for the weight of the displaced air.

Example.—*A solid wooden cylinder, 4 ft. long, floats in water with its axis vertical and a depth of 3 ft. immersed. Find the specific gravity of the wood, taking the specific gravity of air to be* 0·0013.

We have

wt. of cylinder = wt. of water displaced + wt. of air displaced.

Taking s as the specific gravity of the wood and using the weights of the water displaced and the cylinder obtained in Example 1, § 54, this becomes

$$\pi r^2 \cdot 4 \cdot s \cdot 62\tfrac{1}{2} = \pi r^2 \cdot 3 \cdot 62\tfrac{1}{2} + \text{weight of air displaced}$$
$$= \pi r^2 \cdot 3 \cdot 62\tfrac{1}{2} + \pi r^2 \cdot 1 \cdot (0.0013) \cdot 62\tfrac{1}{2}$$
$$4s = 3 + 0.0013,$$
$$s = 0.750325$$

as distinct from the value $s = 0.75$ obtained in the example quoted

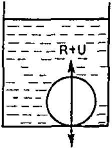


Fig 106

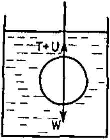


Fig 107

58 Equilibrium of a submerged body

If W denotes the weight of a submerged body, U the upthrust, or force of buoyancy (equal and opposite to the weight of the displaced liquid) then clearly

(i) If $W > U$, the body will sink until it rests on the bottom of the containing vessel If the vessel exerts an upward reaction R on the body, then since the body will be in equilibrium (Fig 106),

$$R + U = W,$$
$$R = W - U$$

Alternatively, there will be equilibrium if the body is supported in a submerged position by a string If the tension in the string is T, then for equilibrium (Fig 107)

$$T + U = W$$
$$T = W - U$$

(ii) If $W = U$, the body will rest in any position providing it is completely submerged (Fig 108)

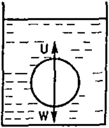


Fig 108

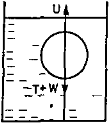


Fig 109

(iii) If $W < U$, the body will rise until it floats on the surface in such a position that the then decreased upthrust exactly balances W

Alternatively, equilibrium may be obtained by constraining the body to remain completely submerged If a string be attached to the body and to the bottom of the vessel (Fig 109) the tension (T) in the string will have to be such that

$$T + W = U, \qquad T = U - W$$

Example 1.—*A cube of gold (specific gravity 19·35), whose edges are 5 cm. in length, is suspended in mercury with 4 cm. of each of its sides submerged. To find the tension in the supporting string.*

$$\text{Volume of cube} = 5 \times 5 \times 5 \text{ c.cm.} = 125 \text{ c.cm.};$$

$$\therefore \quad \text{weight of cube} = 125 \times 19{\cdot}35 \text{ grm.} = 2418{\cdot}75 \text{ grm.};$$

$$\text{volume of mercury displaced} = 5 \times 5 \times 4 \text{ c.cm.} = 100 \text{ c.cm.};$$

$$\therefore \quad \text{weight of mercury displaced} = 100 \times 13{\cdot}6 = 1360 \text{ grm.};$$

$$\therefore \quad \text{required tension of string} = 2418{\cdot}75 - 1360 \text{ grm.}$$

$$= 1058\tfrac{3}{4} \text{ grm. wt.}$$

Example 2.—*If the tension be reduced to 1 kilog., to find how much the cube will sink.*

Here the cube sinks until the weight of the *additional* liquid displaced equals the decrease of tension, or $58\frac{3}{4}$ grm.;

$$\therefore \quad \text{the additional volume displaced} = 58{\cdot}75 \div 13{\cdot}6 \text{ c.cm.} = 4{\cdot}32 \text{ c.cm.}$$

but the area of the base of the cube = 25 sq. cm.;

$$\therefore \quad \text{increase in depth of immersion} = 4{\cdot}32 \div 25 \text{ cm.} = 0{\cdot}1728 \text{ cm.}$$

$$= 1{\cdot}728 \text{ mm.}$$

Example 3.—*To find the weight of a cylindrical cork (specific gravity 0·24) which requires a weight of 13 grm. to sink half the length of its axis in water.*

$$\text{Let the volume of the cylinder} = 2v \text{ c.cm.}$$

$$\text{Then the volume of the water displaced} = v \text{ c.cm.};$$

$$\therefore \quad \text{weight of water displaced} = v \text{ grm.},$$

and $\qquad$ weight of cylinder $= 2v \times 0{\cdot}24$ grm. $= 0{\cdot}48v$ grm.;

therefore, from the equilibrium of the cylinder,

$$0{\cdot}48v + 13 = v;$$

$$\therefore \quad 0{\cdot}52v = 13 \quad \text{or} \quad v = 25 \text{ c.cm.};$$

$$\therefore \quad \text{weight of cork} = 0{\cdot}48v = 12 \text{ grm.}$$

Example 4.—*A solid sphere of mass M and density ρ_1 is fastened below the surface of a liquid of density ρ_2 ($> \rho_1$) by means of a string attached to the lowest point of the sphere and to the bottom of the containing vessel. Find the tension in the string.*

If the liquid slowly drains out of the vessel and $\rho_2 = 3\rho_1$, prove that the tension is reduced to a quarter of its previous value when half the sphere is above the surface. (*Inter. Sc.*)

Assuming the mass of the sphere to be M lb and its density ρ_1 lb per cub ft then if r ft is the radius of the sphere

$$\text{volume of sphere} = \tfrac{4}{3}\pi r^3 \text{ cub ft}$$

$$\text{mass of sphere} = \tfrac{4}{3}\pi r^3 \rho_1 \text{ lb}$$

$$= \text{M lb}$$

$$\text{M} = \tfrac{4}{3}\pi r^3 \rho_1$$

Hence $\qquad \text{weight of sphere} = \tfrac{4}{3}\pi r^3 \rho_1 \text{ lb wt}$

But $\qquad \text{weight of liquid displaced} = \tfrac{4}{3}\pi r^3 \rho_2 \text{ lb wt}$

$$= \frac{\text{M}\rho_2}{\rho_1}$$

If the tension in the string is T lb wt

then tension in string + weight of sphere = weight of liquid displaced

$$\text{T} + \text{M} = \frac{\text{M}\rho_2}{\rho_1}$$

$$\text{T} = \frac{\text{M}(\rho_2 - \rho_1)}{\rho_1} \text{ lb wt}$$

or

$$\text{T} = \frac{\text{M}g(\rho_2 - \rho_1)}{\rho_1} \text{ poundals}$$

If $\rho_2 = 3\rho_1$ this becomes $2\text{M}g$ poundals

When half the sphere is above the surface we have

$$\text{weight of liquid displaced} = \tfrac{1}{2}\,\frac{\text{M}\rho_2}{\rho_1} \text{ lb wt}$$

and if T′ lb wt is the new tension

$$\text{T}' + \text{M} = \tfrac{1}{2}\,\frac{\text{M}\rho_2}{\rho_1}$$

$$= \tfrac{3}{2}\text{M}, \text{ since } \rho_2 = 3\rho_1$$

$$\text{T}' = \tfrac{1}{2}\text{M lb wt}$$

$$= \tfrac{1}{2}\text{M}g \text{ poundals}$$

and this is a quarter of its previous value ($2\text{M}g$ poundals)

59 Effect of immersed solids on pressure

If solids be lowered into a vessel containing liquid the level of the liquid will rise owing to the displacement produced by the solids and therefore there will be an increase of pressure all over the surface of the vessel

Since the pressure at any point of a heavy liquid depends only on the depth and density it follows that the pressure on the sides and bottom of the vessel is the same as if the solids were replaced by liquid equal in amount to that which they displace

Example 1.—Consider a bucket containing water and suspended by a rope. Now, let any body—say a brick—be lowered into the bucket by means of a second rope (Fig. 110). The water will rise in the bucket; there will, therefore, be an increase in the pressure all over the bucket, and the tension in the first rope will be greater than before, since it has to support a greater resultant thrust.

In this case, the tension in the rope supporting the bucket
= weight of bucket + weight of water actually contained in it
+ weight of water displaced by brick.

Also, we know that
tension in rope supporting brick
= weight of brick − weight of water displaced by brick;
∴ sum of tensions in the two ropes
= weight of bucket + actual weight of water + weight of brick,

as evidently should be the case, for the two ropes together have to support the bucket, the water and the brick.

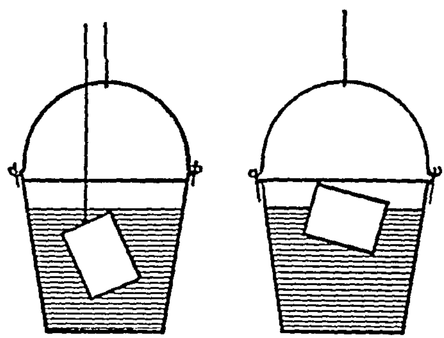


Fig. 110. Fig. 111.

Example 2.—If, instead, we place in the bucket a body lighter than water—say a block of wood—and allow it to float (Fig. 111), it will displace a quantity of water of weight equal to its own weight. As before, we have

tension in supporting rope
= weight of bucket + weight of water actually contained in it
+ weight of water displaced by wood
= weight of bucket + weight of contained water
+ weight of wood;

as evidently should be the case, since the rope has to support the bucket, the water and the wood.

Example 3.—*A bucket containing water is suspended by a rope which passes over a smooth pulley, and is balanced by a weight* W_1 *at the other end of the rope. A piece of wood of weight* W_2 *and specific gravity* σ *is then fastened to the bottom of the bucket by a string, so that it is completely immersed. Prove that, during the ensuing motion, the tension in the string attached to the wood is*

$$\frac{2W_1W_2}{2W_1 + W_2}\left(\frac{1-\sigma}{\sigma}\right). \qquad \textit{(Inter. Sc.)}$$

When the piece of wood of weight W_2 is fastened to the bucket we have a total weight of $W_1 + W_2$ on this side of the pulley compared with the weight W_1 on the other side of the pulley

Hence if the bucket moves down with an acceleration f this is given by

$$f = \frac{W_2 g}{2W_1 + W_2} \qquad \text{(i)}$$

(see *Tutorial Dynamics* § 123)

If T lb wt is the tension in the string which attaches the piece of wood to the bucket and if V lb wt is the resultant upward thrust exerted by the liquid on the piece of wood then the forces acting on the wood are

(1) T and W_2 acting downwards

(2) V acting upwards

Since this piece of wood also moves downwards with an acceleration f (since it is part of the bucket and contents) the equation of motion for the piece of wood is

$$(T + W_2 - V)\,g = W_2 f \qquad \text{(ii)}$$

The weight of the piece of wood is W_2 lb wt and its specific gravity is σ so that the weight of an equal volume of water is $\frac{W_2}{\sigma}$ lb wt If the wood were replaced by this equal volume of water the weight of this water together with the upthrust V would give this mass of water an acceleration f downwards hence

$$\left(\frac{W_2}{\sigma} - V\right) g = \frac{W_2}{\sigma} f \qquad \text{(iii)}$$

We require to find T from these three equations Subtract (iii) from (ii)

$$\left(T + W_2 - \frac{W}{\sigma}\right) g = W_2 f\left(1 - \frac{1}{\sigma}\right)$$

$$Tg - W_2\left(1 - \frac{1}{\sigma}\right)(f - g)$$

$$W_2\left(1 - \frac{1}{\sigma}\right)\left(\frac{-2W_1 g}{2W_1 + W_2}\right) \text{ from (i)}$$

$$= \frac{2W_1 W_2 g}{2W_1 + W_2}\left(\frac{1}{\sigma} - 1\right)$$

$$T = \frac{2W_1 W_2}{2W_1 + W_2}\left(\frac{1 - \sigma}{\sigma}\right)$$

60 Equilibrium of bodies floating under constraint

If a body is floating in a liquid not freely but with one point fixed we may easily establish the relation which must exist between the

forces and the positions of their lines of action in order that the body shall be maintained in equilibrium.

Let Fig. 112 represent the body free to turn about the fixed point O. G is the centre of gravity of the body and H is the centre of gravity of the displaced liquid (*i.e.* the centre of buoyancy).

Consider the forces acting on the body. There is no resultant horizontal force on the body (see § 52). The only forces on the body are

(1) The weight of the body (W) acting vertically downwards through its centre of gravity (G).

(2) The force of buoyancy (or upthrust), U, acting vertically upwards through the centre of buoyancy (H).

(3) The reaction (R) exerted by the support at the fixed point O.

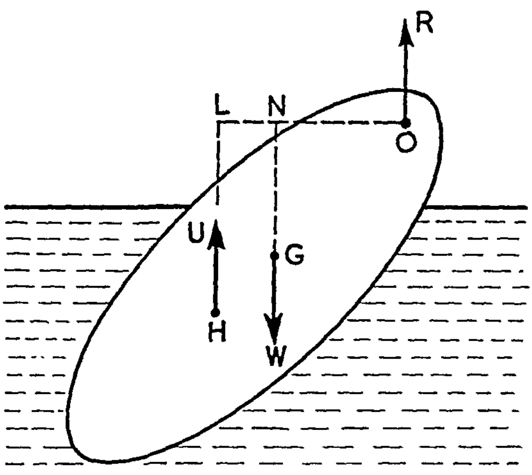


Fig. 112.

Draw a horizontal line through O, meeting the lines of action of W and U in N and L respectively. Then, for equilibrium, since U and W act vertically, so also must R, and the magnitudes of these forces must be such that

$$R + U = W.$$

Also, they must all lie in the same vertical plane and the moment of W about O must balance the moment of U about O, *i.e.*

$$W \times ON = U \times OL.$$

A particular case will be when O, G, H are all in the same vertical line.

Example 1.—*A uniform heavy rod a foot long is freely movable round a horizontal axis fixed at one extremity at a height of 2 in. above the*

surface of the water in which the rod partly rests, find the position of equilibrium, and if the position is oblique, show that the specific gravity of the rod must be less than $\frac{35}{36}$

Let AB (Fig 113) represent the rod, A being the fixed end and let C be the point where the surface of the water meets the rod If $AC = x$ in, then $BC = 12 - x$ in G is the mid point of the rod and is therefore its centre of gravity since the rod is uniform,

$$AG = 6 \text{ in}$$

H is the mid point of BC and is the centre of gravity of the displaced water,

$$CH = \tfrac{1}{2}(12 - x) \text{ in}$$

If the cross sectional area of the rod is a sq in, and its specific gravity is s, then

$$\text{volume of rod} = 12a \text{ cub in},$$

$$\text{weight of rod} = 12asw \text{ lb wt},$$

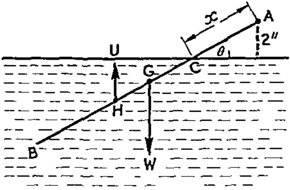


Fig 113

where w = weight in pounds of 1 cub in of water,

$$W = 12asw \qquad \text{(i)}$$

The volume of water displaced is the volume of the immersed part of the rod (i e the length BC),

$$\text{volume of water displaced} = (12 - x)\,a \text{ cub in},$$

$$\text{weight of water displaced} = (12 - x)\,aw \text{ lb wt},$$

$$U = (12 - x)\,aw \qquad \text{(ii)}$$

If the rod makes an angle θ with the horizontal taking moments about A,

$$W \times AG \cos\theta = U \times AH \cos\theta,$$

$$W \times AG = U \times AH \qquad \text{(iii)}$$

but $AG = 6$ in and $AH = x + \frac{1}{2}(12 - x) = \frac{1}{2}(12 + x)$ in,

$\therefore$ (iii) becomes $$6W = \tfrac{1}{2}(12 + x)\,U,$$
or, substituting for W and U,
$$6 \cdot 12asw = \tfrac{1}{2}(12 + x)(12 - x)\,aw,$$
i.e. $$144s = 144 - x^2;$$
$$\therefore\quad x^2 = 144 - 144s \quad \text{................(iv)}$$
which gives the length of rod not immersed, in terms of the specific gravity of the rod.

For the oblique position of equilibrium, AC must be greater than 2 in., *i.e.* $x > 2$;
$$\therefore\quad x^2 > 4;$$
$$\therefore\quad 144 - 144s > 4 \text{ from (iv)};$$
$$\therefore\quad 144s < 140;$$
$$\therefore\quad s < \tfrac{35}{36}.$$

Example 2.—ABCD, ABEF *are respectively horizontal and vertical faces of a uniform rectangular block of wood of weight* W *floating in water,* AB *being an upper horizontal edge. When a rope is fastened to the mid-point of* AB *and pulled vertically upwards until* CD *is in the surface of the water it is found that* EF *is also in the surface. Prove that the specific gravity of the wood is* 2/3 *and find the tension in the rope.*

(*Inter. Sc.*)

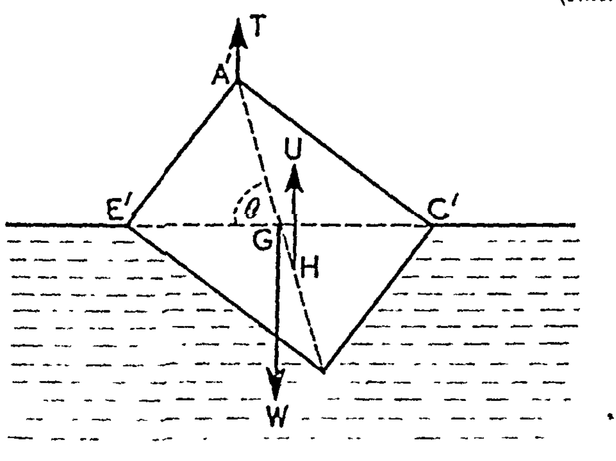


Fig. 114.

Let Fig. 114 represent the vertical section through the mid-point of the horizontal edges of the block of wood when both CD and EF are in the surface, *i.e.* A′, C′, E′ are the mid-points of AB, CD, EF respectively. G is the centre of gravity of the block of wood and is therefore at the point of intersection of the diagonals of this middle

section H is the centre of buoyancy Its position is determined by the fact that the water displaced is in the shape of a triangular prism H is the centre of gravity of this prism and is such that

$$GH = \tfrac{1}{3} \times \text{the semi diagonal}$$
$$= \tfrac{1}{3}A\,G$$
$$A\,H = \tfrac{4}{3}A\,G \qquad \text{(i)}$$

If T is the tension in the rope attached to A and U the force of buoyancy, then

$$T + U = W \qquad \text{(ii)}$$

and taking moments about A,

$$W \times A\,G \cos\theta = U \times A'H \cos\theta \qquad \text{(iii)}$$

where θ is the angle made by A G with the horizontal

The weight of half the block of wood is $\frac{1}{2}$W, therefore the weight of water displaced by the immersed half of the block is $\frac{1}{2}\,\frac{W}{\sigma}$ where σ is the specific gravity of the wood *i e*

$$U = \frac{W}{2\sigma} \qquad \text{(iv)}$$

Substituting the value of A'H from (i) in (iii) and cancelling $\cos\theta$,

$$W \times A\,G = U \times \tfrac{4}{3}A\,G,$$
$$W = \tfrac{4}{3}U$$
$$= \frac{4}{3}\,\frac{W}{2\sigma} \text{ from (iv)}$$
$$6\sigma = 4$$
$$\sigma = \tfrac{2}{3} \qquad \text{(v)}$$

and from (ii)

$$T = W - U$$
$$= W - \frac{W}{2\sigma} \text{ from (iv)}$$
$$= W - \frac{3W}{4} \text{ from (v)}$$
$$= \tfrac{1}{4}W$$

Example 3 —*A thin uniform rod AB of length a and specific gravity s_1 (<1) is suspended by a string attached to A When the end B is immersed in water a length b of the rod remains out of the water, and when immersed in a liquid of specific gravity s_2 a length c of the rod is out of the liquid, prove that*

$$s_1 = \frac{a^2 - b^2}{a^2} \quad s_2 = \frac{a^2 - b^2}{a^2 - c^2} \qquad \textit{(Inter Sc)}$$

Consider Fig. 115. C is the point of intersection of the rod with the surface of the water so that $AC = b$. G is the centre of gravity of the rod, *i.e.* $AG = \frac{1}{2}a$. H is the centre of buoyancy, so that

$$CH = \tfrac{1}{2}BC = \tfrac{1}{2}(a - b);$$
$$\therefore \quad AH = b + \tfrac{1}{2}(a - b)$$
$$= \tfrac{1}{2}(a + b).$$

If the weight of water displaced by unit length of the rod is w,

the weight of the rod $= as_1w$,

and force of buoyancy $= (a - b)w$.

Taking moments about A,

$$as_1w \times AG \cos\theta = (a - b)w \times AH \cos\theta,$$

where θ is the angle made by rod with the horizontal;

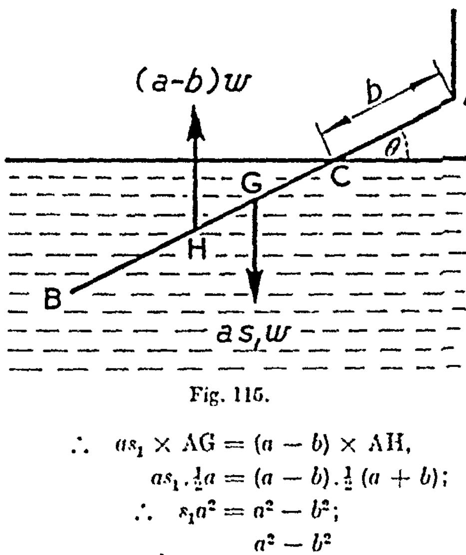


Fig. 115.

$$\therefore \quad as_1 \times AG = (a - b) \times AH,$$
$$as_1 . \tfrac{1}{2}a = (a - b) . \tfrac{1}{2}(a + b);$$
$$\therefore \quad s_1a^2 = a^2 - b^2;$$
$$\therefore \quad s_1 = \frac{a^2 - b^2}{a^2}.$$

Let Fig. 116 represent the rod when immersed in a liquid of specific gravity s_2. Suppose the rod now makes an angle ϕ with the horizontal and C′ is the point of intersection of the rod with the surface of the liquid. If H′ is the new centre of buoyancy, $C'H' = \frac{1}{2}BC' = \frac{1}{2}(a - c)$ so that

$$AH' = c + \tfrac{1}{2}(a - c) = \tfrac{1}{2}(a + c).$$

As before, $AG = \frac{1}{2}a$ and the weight of the rod is as_1w. The length of rod immersed is $a - c$, so that the weight of liquid displaced is $(a - c)s_2w$ and this is the force of buoyancy.

Again, taking moments about A,

$$as_1w \times AG\cos\phi = (a-c)\,s_2w \times AH'\cos\phi,$$

$$\therefore \quad as_1w \times \tfrac{1}{2}a = (a-c)\,s_2w \times \tfrac{1}{2}(a+c);$$

$$\therefore \quad s_1a^2 = s_2(a^2-c^2),$$

$$\therefore \quad s_2 = \frac{a^2}{a^2-c^2}\,s_1$$

$$= \frac{a^2}{a^2-c^2}\,\frac{a^2-b^2}{a^2}$$

from the previous result,

$$= \frac{a^2-b^2}{a^2-c^2}$$

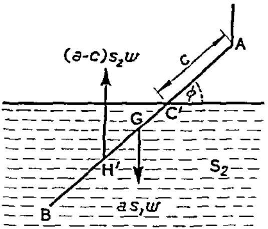


Fig 116

Exercises VIII

(1) A thin uniform wooden rod, 1 ft in length, floats vertically in water with 2 in above the surface Find the specific gravity of the wood

(2) A buoy weighs 24 lb What is its volume if it floats with three-quarters of its volume immersed in sea water weighing 64 lb per cub ft

(3) Prove that the buoyancy of a life-buoy made of cork of specific gravity s, weighing w lb., is $w\left(\frac{1}{s}-1\right)$ lb. wt

(4) A ship passes from sea water (sp gr 1·024) to fresh water Does it rise or sink deeper into the water?

By how much does the water line shift if the ship weighs W ton and the area of the water line section (assumed constant over the range of variation) is A sq ft (T[illegible] -ght of 1 cub ft of fresh water is 62 5 lb) [illegible] prove that [illegible] more than six inches for a ship weighing $s_1 = \frac{a^2-b^2}{a^2}$, $s_2 =$ section must have an area greater

(5) A block of wood (specific gravity 0·75), whose volume is 250 c.cm., is totally immersed in a liquid of specific gravity 1·25 by means of a string attached to the bottom of the vessel containing the liquid. Find the tension of the string.

(6) A vessel containing water is placed in one scale of a balance and counter-balanced by weights. A person dips his hand in without touching the sides of the vessel. Will the equilibrium be disturbed? Give your reasons.

(7) A cylinder of uniform cross-section is made up of a length L of a substance of specific gravity D and a length l of another substance of specific gravity d; it floats vertically in water. Determine how much of it is submerged.

(8) A cylinder of wood, 2 ft. long and of weight W, floats with its axis vertical in a fluid of three times its own specific gravity. What forces must be applied to the cylinder to keep it (a) 2 in. above, and (b) 2 in. below the equilibrium position?

(9) A solid cylinder of specific gravity 0·7 floats with its axis vertical in a vessel containing two liquids whose specific gravities are 0·6 and 0·9, the cylinder being completely submerged. How much of its axis is in the upper fluid?

(10) A uniform rod rests half immersed in water in an oblique position and can rotate in a vertical plane about a point distant one-sixth of its length from the end below the water. Find the specific gravity of the rod.

(11) A block of wood, whose weight is 63 lb. and whose specific gravity is 0·6, is in a pond. If a ball of lead, whose specific gravity is 11·5, is attached to the block by a string, find the least weight which the ball can have so as to keep the block quite under water.

(12) A cube of wood floating in water supports a weight of 480 oz. On the weight being removed, it rises 1 in. Find the size of the cube.

(13) A right cone, whose weight is W, floats in a liquid, vertex downwards, with one-third of its axis immersed. What additional weight must be placed on the base of the cone so as just to sink it entirely in the liquid?

(14) A thin uniform rod of weight W floats in water in an inclined position with one-eighth of its length out of the water when a weight P of negligible volume is attached to the lower end. Show that $7P = W$.

(15) A cylinder of wood floats in water with its axis vertical and having three-fourths of its length immersed. Oil whose weight is half that of water is then poured into the vessel to a sufficient depth to cover the wood. How much of the cylinder will now be immersed in water?

(16) A body of specific gravity s is completely immersed in liquid of specific gravity ρ. Show that its acceleration downwards is $\left(1 - \frac{\rho}{s}\right) g$.

(17) A solid cone of height h and density ρ floats, with its vertex upwards, in a liquid of density σ ($> \rho$). How much of its axis is above the surface of the liquid?

(18) A right prism of triangular cross section AOB, made from uniform material, floats in water with the edge through O in the surface and the face containing OB out of the water Prove that the face containing AB will be vertical

If OA = 4 cm, AB = 5 cm, and the angle AOB is a right angle, find the specific gravity of the material of the prism (H S C, I)

(19) A cube whose specific gravity is 0 8 rests in a rectangular tank with a square bottom An edge of the cube is 15 in, and a side of the square is 16 in Just enough water is poured in to float the cube Compare the volume of water with that of the cube

(20) A uniform thin rod AB is free to turn about A which is fixed at a height $\frac{1}{4}$AB above the surface of water into which the rod dips If the rod is made of wood of specific gravity 0 36, find what fraction of the rod will be under water in the oblique position of equilibrium (H S C, I)

(21) A solid homogeneous cone floats with its axis vertical and vertex uppermost in a liquid whose density is to that of the cone as 125 to 61, how much of the axis is immersed?

(22) A solid is held completely immersed in a cylindrical vessel containing water by means of a string whose tension is T attached to the base of the vessel, the axis of which is vertical The string is cut, and the solid comes to rest floating in the water Prove that the surface of the water falls a distance $T/\pi r^2 w$, where r is the radius of the vessel, and w the weight of unit volume of water (Inter Sc)

(23) A ship passing from the sea into fresh water sinks m inches and then on discharging P ton of cargo rises n inches Assuming the sides of the ship to be vertical near the water line, and that the specific gravity of sea water is 1 026 prove that the weight of the ship and cargo was originally $513m\mathrm{P}/(13n)$

(24) A mass composed partly of solid copper (sp gr 8 8) and partly of solid lead (sp gr 11 4), floats with two thirds of its bulk immersed in mercury (sp gr 13 6), and the remaining one third in water Compare the volumes and weights of the copper and lead in the mass

(25) A thin rod of density ρ has a small piece of metal of weight $\frac{1}{n}$th that of the rod attached to one end Prove that the rod will float at any inclination in a liquid of density σ if $\frac{\sigma}{\rho} = \left(1 + \frac{1}{n}\right)^2$

(26) AB is a uniform rod floating in water, and its specific gravity is 0 5, a string is fastened to the end A, by which A is lifted to a height above the surface of the water equal to a quarter of the length of the rod Find the position in which the rod comes to rest

(27) A solid body is suspended by a thread, and submerged successively in three liquids The tension of the thread diminishes in the change from the second to the third liquid by just as much as it diminishes from the first to the second Find the relation between the specific gravities of the liquids

(28) A flexible cord passes over a pulley which is fixed above the surface of water. One end of the cord is held in the hand and the other is attached to one end of a uniform wooden pole of length $2a$ and sp. gr. s, resting partly immersed. Show that, however the cord is pulled, the length of the immersed part of the pole remains constant until a certain position is reached.

(29) A body floating in a liquid has volumes V_1, V_2, V_3 above the surface when the air density is σ_1, σ_2, σ_3 respectively. Prove that

$$\frac{\sigma_2 - \sigma_3}{V_1} + \frac{\sigma_3 - \sigma_1}{V_2} + \frac{\sigma_1 - \sigma_2}{V_3} = 0. \qquad \text{(H.S.C., I.)}$$

(30) A slab whose density is s floats wholly immersed in a vessel containing liquids of density s_1 and s_2. If s lies between s_1 and s_2, find in what ratio the thickness of the slab is divided by the surface of separation. If h is the thickness of the slab, and d the depth of the upper fluid, what condition must be fulfilled in order that the slab may float partially immersed.

(31) A buoy, in the form of a right circular cone, the vertex of which is fastened to the bottom of the sea by a chain, floats at low tide with the chain slack, the axis of the cone being vertical and two-thirds of it immersed. If the diameter of the base is 9 ft. and the height 12 ft., calculate the weight of the buoy assuming that the weight of the water is 64 lb. per cub. ft. and that the weight of the chain is negligible.

Calculate also the pull on the chain at high tide when the buoy is completely immersed. (Inter. Eng.)

(32) A solid hemisphere of wood, whose specific gravity is $\frac{2}{3}$, has a solid right cone of the same wood constructed on its base, the bases being wholly coincident. If r is the radius of the hemisphere, and $2r$ the height of the cone, and the compound body floats in water with the axis of the cone vertical, will the hemispherical surface be wholly, or only partially, submerged? Calculate the position.

(33) A uniform straight rod of small cross-section, length a, weight W, made of material of specific gravity σ (<1), is freely hinged to the bottom of a vessel, of depth greater than a, so that it can turn freely in a vertical plane. Initially the rod rests horizontally on the bottom and water is slowly poured in. Investigate the change of length of rod immersed and of the reaction at the hinge as the depth of water increases, illustrating your results graphically.

If, when the rod is inclined at an acute angle to the horizontal, a second thin uniform rod, of material of specific gravity λ, is similarly freely hinged to the bottom of the vessel and is in equilibrium inclined at the same angle to the horizontal, find the length of the second rod.

(H.S.C., III.)

(34) Draw a semicircular arc ACB, of which C is the middle point, and let it represent a weightless hemispherical bowl; a particle is fixed at C, and an equally heavy particle is fixed to A; when the bowl is floated in water it is found that A just comes to the surface of the water; if now the particle is removed from A and put into the bowl, find the height of AB above the surface of the water.

(35) A bucket half full of water is suspended by a string which passes over a smooth pulley small enough to let the other end fall into the bucket To this end is tied a ball whose sp gr is greater than 2 Show that, for equilibrium, if the ball does not touch the bottom of the bucket and if no water overflows, the weight of the ball equals $sW/(s-2)$ where W is the weight of the bucket and water

(36) A solid cone is floating in water with its axis vertical and vertex downwards To cause it to sink until $\frac{3}{4}$ of its axis is immersed requires a load of 50 grm on its base, and to cause $\frac{4}{5}$ of the axis to be immersed requires a load of 96 grm Show that the specific gravity of the body is very nearly 0 324

(37) A uniform rod of length l and specific gravity ρ is free to turn about one end which is at a distance $h\ (<l)$ above the surface of separation of two liquids which do not mix, whose specific gravities are σ_1 and σ_2 $(\sigma_1<\sigma_2)$ The rod rests in equilibrium at an inclination θ to the vertical with its free end in the lower liquid Show that

$$\cos^2\theta = \frac{h^2(\sigma_2-\sigma_1)}{l^2(\sigma_2-\rho)}$$

and that $\sigma_1 < \rho < \sigma_2$ (H S C, III)

(38) Show that the upthrust on a solid in a liquid, the solid and liquid moving together with a common acceleration f vertically downward, is

$$W\left(1-\frac{f}{g}\right)$$

where W is the weight of the liquid which would occupy the space filled by the solid

A bucket containing water is suspended at one end of a light inextensible string passing over a fixed smooth pulley and is in equilibrium with a counterpoise of mass M at the other end of the string A solid of mass m and specific gravity $\sigma\ (<1)$ is then attached inside to the bottom of the bucket by a string which keeps the solid completely immersed When the system is released, show that the ratio of the tensions in the two strings is $(M+m)\,\sigma/m\,(1-\sigma)$ (H S C, III)

(39) A uniform cone of given weight floats in a liquid with its vertex downwards Prove that the surface of the cone in contact with the liquid is least when its semi vertical angle is $\tan^{-1}(1/\sqrt{2})$

(40) A uniform cylindrical body of horizontal cross sectional area mA floats with its axis vertical, partly immersed in liquid of density ρ contained in a cylindrical vessel of horizontal cross sectional area nA $(n>m)$ A volume pAa of a lighter liquid of density ρ' is gently poured into the vessel and the liquids do not mix The body continues to float partly immersed in both liquids, prove that the height through which the body rises in the vessel is $\frac{p}{n}\frac{\rho'}{\rho}a$ (H S C, I)

ANSWERS

1. $\frac{5}{6}$. 2. $\frac{1}{2}$ cub. ft. 4. Sinks, 0·84 M/A ft. 5. 125 grm. wt.

6. Yes. The scale pan having the vessel will go down, for the level of the water is raised, and consequently the pressure on the base is increased.

7. $ld + LD$. 8. (a) $\frac{3}{4}$W; (b) $\frac{1}{4}$W. 9. $\frac{2}{3}$. 10. $\frac{1}{8}$.

11. 46 lb. wt. 12. Side is 28·8 in. 13. 26W. 15. $\frac{1}{2}$ volume.

17. $h\left(1 - \dfrac{\rho}{\sigma}\right)^{\frac{1}{3}}$. 18. $\frac{19}{25}$. 19. 0·11 : 1. 20. $\frac{1}{5}$.

21. $\frac{1}{5}$. 24. Volumes 10 : 3; weights 440 : 171.

26. At angle $\sin^{-1}\{1/(2\sqrt{2})\}$ with horizontal.

27. $s_1 - 2s_2 + s_3 = 0$. 30. $(s_2 - s) : (s - s_1)$; $d < h(s_2 - s)/(s_2 - s_1)$.

31. 1536π lb. wt., 3648π lb. wt.

32. Wholly. The common base will be at a depth $2r\left(1 - \sqrt[3]{\frac{3}{5}}\right)$.

33. Length $= a(\sigma/\lambda)^{\frac{1}{2}}$. 34. $\frac{1}{2}\sqrt{2}$ times the radius.

CHAPTER IX

THE PRACTICAL DETERMINATION OF SPECIFIC GRAVITY

61 Introduction

In this chapter we propose to consider various practical methods of determining the specific gravity of solids and liquids The problem is of importance commercially, since the specific gravity of a liquid is frequently taken as a measure of its purity Again when a liquid is diluted with water, the degree of dilution is ascertainable from the specific gravity of the mixture

If we require to find the specific gravity (s) of a solid whose volume (V cub ft) may be calculated by simple mensuration, this may be obtained directly, if the weight (W lb wt) is found, since

$$W = Vsw,$$

where $w =$ weight of 1 cub ft of the standard substance (water), *i e* $w = 62\ 3$ lb wt

When the specific gravity is not directly obtainable we shall use one of the following methods —

(i) The Specific Gravity bottle,

(ii) The Hydrostatic Balance,

(iii) Hydrometers, or

(iv) The U tube

We shall consider each of these in order, but before doing so, we give a list of the principal specific gravities

LIQUIDS

Acid (Hydrochloric)	1 19	Olive Oil	0 92
,, (Nitric)	1 27	Petrol	0 70
,, (Sulphuric)	1 84	Turpentine	0 88
Alcohol	0 83	Water (Pure)	1 00
Glycerine	1 26	,, (Sea)	1 03
Milk	1 03		

SOLIDS

Asbestos	3 00	Porcelain	2 14
Copper sulphate crystals	2 38	Quartz crystals	2 65
Diamond	3 50	Salt (common)	1 92
Glass	2 50-3 50	Sugar	1 60

Solids (*continued*)

Granite	2·64	Wood:	
Ice	0·92	Alder	0·54
Marble	2·70	Ash	0·75
Metals:		Beech	0·85
Aluminium	2·65	Box	1·04
Antimony	6·70	Cedar	0·52
Bismuth	9·80	Cherry	0·79
Brass	8·40	Cork	0·24
Copper	8·88	Ebony	1·33
Gold	19·26	Elm	0·57
Iron	7·80	Mahogany	0·90
Lead	11·35	Oak	0·70-0·95
Mercury	13·59	Pine	0·53
Platinum	21·10	Poplar	0·38
Silver	10·70	Willow	0·59
Tin	7·29		
Zinc	7·20		

62. Specific gravity bottle

The specific gravity bottle is much used for finding the specific gravities of solids and liquids. It is constructed for the purpose of weighing exactly equal volumes of different liquids, and it consists of a glass flask having a tightly fitting stopper through which a very fine hole (*ab*) is bored (Fig. 117). In using the bottle, it is completely filled with the liquid to be weighed, and the stopper is then pushed in till it reaches a certain mark (P) on the neck of the bottle. The superfluous liquid overflows through the hole *ab*, and is wiped off; so the bottle, when filled in this way, always contains the same volume of liquid.

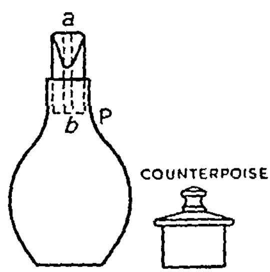


Fig. 117.

To obviate the necessity of allowing for the weight of the bottle in every observation, a counterpoise is provided, whose weight is exactly equal to that of the bottle. This counterpoise is usually a little metal case containing small shot, and its weight is adjustable by adding or subtracting shot.

When the bottle, filled with liquid, is placed in one of the scale-pans of a balance, the counterpoise is placed in the other pan in addition to the weights used in weighing. Since the counterpoise balances the weight of the bottle, the additional weights give the weight of the contained liquid alone.

(*a*) *To find the specific gravity of a given liquid*

The procedure is as follows —

(i) Adjust the weight of the counterpoise (if necessary) until it balances the bottle when empty

(ii) Fill the bottle with water, carefully insert the stopper and weigh, placing the counterpoise in the scale pan containing the weights

(iii) Fill the bottle with the liquid whose specific gravity is required, carefully insert the stopper, and again weigh, as before

The second process gives the weight of the water contained in the bottle The third process gives the weight of an equal volume of the given liquid Dividing the latter by the former, the specific gravity of the liquid is found

If no counterpoise is available let w grm be the weight of the empty bottle Then if it weighs W grm and W′ grm when filled with water and the given liquid respectively, we have

$$\text{weight of water in bottle} = W - w,$$

$$\text{weight of equal volume of liquid} = W' - w$$

$$\text{specific gravity of liquid} = \frac{W' - w}{W - w}$$

Example —*A flask weighs 7 2 grm when empty 53 45 grm when filled with sulphuric acid and 32 2 grm when filled with water To find the specific gravity of sulphuric acid*

$$\text{Weight of sulphuric acid} = 53\ 45 - 7\ 2\text{ grm} = 46\ 25\text{ grm},$$

$$\text{weight of equal volume of water} = 32\ 2 - 7\ 2\text{ grm} = 25\ 0\text{ grm},$$

$$\text{specific gravity of sulphuric acid} = 46\ 25 - 25 = 1\ 85$$

(*b*) *To find the specific gravity of a solid insoluble in water*

If a solid can be broken up into small enough pieces to go into a specific gravity bottle, then we may find its specific gravity as follows —

(i) Weigh the solid (w grm)

(ii) Fill the specific gravity bottle with water, and place it together with the solid in one of the scale pans of a balance and weigh Let this weight, *i e* the weight of the bottle full of water + the weight of the solid, be W_1 grm

(iii) Take the solid and insert it in the bottle A quantity of water will overflow whose volume is equal to that of the solid and the volume of water in the bottle will be less than before by the volume of the solid If, therefore, the bottle containing the solid and water be again weighed (W_2 grm) their total weight will be less than before by the weight of the displaced water, *i e*

$$\text{weight of water equal in volume to that of the solid} = W_1 - W_2$$

$$\text{But specific gravity of solid} = \frac{\text{weight of solid}}{\text{weight of equal volume of water}}$$

$$= \frac{w}{W_1 - W_2}.$$

Suitable solids which may be used for this experiment are sand, copper filings, pieces of marble, iron, lead, zinc or brass, or quartz crystals.

Example 1.—*The weight of a solid is 13 grm. When the specific gravity bottle is filled with water, its weight, together with that of the solid, is 63 grm. When the solid is put into the bottle, the combined weight is 53 grm. To find the specific gravity of the solid.*

After the solid is dropped into the bottle, the volume of water in the bottle is less than it was before by an amount equal to the volume of the solid.

Hence the difference of weights, 63 − 53 or 10 grm., equals the weight of a quantity of water equal in volume to the solid.

But the weight of the solid is 13 grm.;

$$\therefore \quad \text{specific gravity of solid} = \tfrac{13}{10} = 1{\cdot}3.$$

Example 2.—*The weight of a quantity of powder (insoluble in water) is p. The weight of a specific gravity bottle filled with water is A, and when the bottle contains the powder and is filled with water its total weight is B. To find the specific gravity of the powder.*

Let w be the weight of water whose volume is equal to that of the powder.

Then, before the powder is placed in the bottle, the total weight of the powder, bottle, and water $= p + \text{A}$.

When the powder is placed in the flask it displaces a quantity of water equal in volume to the powder, whose weight is w.

Therefore the total weight B is less than before by w, that is,

$$\text{B} = p + \text{A} - w;$$

$$\therefore \quad w = p + \text{A} - \text{B};$$

$$\therefore \quad \text{sp. gr. of powder} = \frac{\text{weight of powder}}{\text{weight of water displaced}} = \frac{p}{p + \text{A} - \text{B}}.$$

Hence, if p, A, and B are known, the specific gravity can be found. Notice that it is not necessary to know the weight of the specific gravity bottle itself.

(c) *To find the specific gravity of a solid soluble in water.*

We may use the specific gravity bottle to find the specific gravity of a solid which is soluble in water, provided that we can find a liquid

in which the solid is insoluble The specific gravity of this liquid must be known or found as in (a), let this specific gravity be s If we now carry through procedure (b) using *the liquid in which the solid is insoluble in place of water*, we have the three weighings w, W_1 and W_2 as before Then $W_1 - W_2$ is now the weight of the *liquid* equal in volume to that of the solid,

$$\therefore \text{ wt of an equal volume of } water = \frac{W_1 - W_2}{s},$$

$$\therefore \text{ specific gravity of solid} = \frac{\text{wt of solid}}{\text{wt of equal volume of water}}$$

$$= \frac{w}{(W_1 - W_2)/s}$$

$$= \frac{ws}{W_1 - W_2}$$

Sugar is insoluble in turpentine and so its specific gravity may be obtained by this method

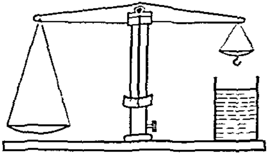


Fig 118

63. The hydrostatic balance

When a common balance is adapted for weighing bodies suspended in fluid, it is called a *Hydrostatic Balance* (Fig 118) The only difference between a hydrostatic balance and an ordinary pair of scales is that one of the scale pans in the former is at a sufficient height to allow a vessel of fluid to be placed under it, and has a hook on its under side from which any small solid may be suspended by means of a fine wire, and weighed when immersed in the fluid.

(a) *To find the specific gravity of a solid which would sink in water* (i e *specific gravity* > 1)

The Hydrostatic Balance may here be used as follows —

(i) Place the solid in the scale-pan and weigh (W)

(ii) Let the solid now be suspended in water by a very fine thread attached to the scale-pan of the balance (Fig 119) and weigh again (P)

From Archimedes' Principle, the difference of the observed weights in air and water, i e $W - P$, is the weight of a quantity of water equal in volume to the solid Thus, $W - P$ = weight of water displaced by solid

Therefore,

$$\text{specific gravity of solid} = \frac{\text{weight of solid}}{\text{weight of water displaced}}$$
$$= \frac{W}{W - P}.$$

Example 1.—*A solid weighs* 15 *grm. in air and* 5 *grm. in water; to find its specific gravity.*

The weight in water is less than in air by the weight of the water displaced;

$\therefore$ weight of water displaced $= 15 - 5 = 10$ grm.

Also weight of solid $= 15$ grm.;

$\therefore$ specific gravity of solid $= \frac{15}{10} = 1{\cdot}5$.

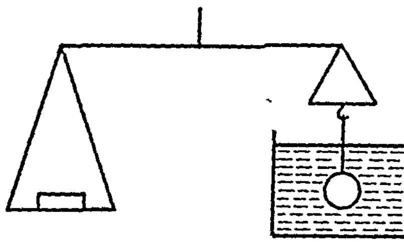


Fig. 119.

Example 2.—*A piece of gold weighs* 598·3 *grm. in air, and* 567·3 *grm. in water; to find its volume and specific gravity.*

Weight of water displaced $= 598{\cdot}3 - 567{\cdot}3$ grm. $= 31$ grm.;

$\therefore$ volume of gold $= 31$ c.cm.,

and specific gravity of gold $= 598{\cdot}2 \div 31 = 19{\cdot}3$.

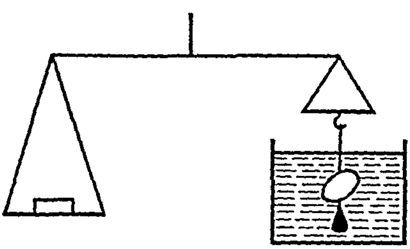


Fig. 120.

(*b*) *To find the specific gravity of a solid which would float in water* (i.e. *specific gravity* < 1).

If the solid were suspended by itself, it would float in water, and we could not find the weight of a quantity of water equal in volume to the whole solid. To remedy this, a heavy piece of metal, called a *sinker*, is attached to the thread which supports the body to be weighed, and this keeps the body under water.

The operations are best performed in the following manner:—

(i) Weigh the solid in air.

(ii) Suspend the solid and sinker together from the scale-pan of the balance, and weigh them in water (Fig. 120).

(iii) Weigh the sinker in water.

The weight in water of the solid and sinker combined is less than the weight in water of the sinker alone by the amount of the total resultant upward force on the solid. This force is the excess of the weight of water displaced by the solid over the weight of the solid itself. This being known and the weight of the solid being also known from the first observation, the weight of the water displaced is found and the specific gravity of the solid can be found, as before.

Example.—*A solid weighs 16 grm in air. When attached to a sinker and immersed in water, the two together weigh 6 grm. The weight of the sinker in water alone is 10 grm. To find the specific gravity of the solid.*

Here the weight of the solid and sinker in water together
= (weight of sinker in water) + (weight of solid in air)
− (weight of water displaced by solid)
= 6 grm.

But weight of sinker in water = 10 grm.

and weight of solid in air = 16 grm.

∴ weight of water displaced by solid = 16 + 10 − 6 grm.
= 20 grm.

$$\text{specific gravity of solid} = \frac{\text{weight of solid}}{\text{weight of water displaced}} = \tfrac{16}{20} = 0{\cdot}8$$

Algebraically, we may represent the result as follows:—

Let W = weight of solid in air,
B = weight of solid and sinker in water together
and A = weight of sinker in water.

Since weight of solid and sinker in water together
= (weight of sinker in water) + (weight of solid in air)
− (weight of water displaced by solid),

$B = A + W -$ (weight of water displaced by solid)

weight of water displaced by solid $= A + W - B$

$$\text{But}\quad \text{specific gravity of solid} = \frac{\text{weight of solid}}{\text{weight of water displaced}}$$

$$= \frac{W}{W + A - B}$$

(c) *To find the specific gravity of a given liquid.*

This may be found by means of the Hydrostatic Balance as follows:—Take any solid body of greater specific gravity than either

the liquid or water (the sinker used in the experiments of the last section is convenient) and

(i) weigh the solid in air (W),

(ii) weigh it in the given liquid whose specific gravity is required. Let this weight be Q, and

(iii) weigh it in water (P).

The difference between the weights in air and in the given liquid (*i.e.* W — Q) is the weight of the liquid displaced by the solid. The difference between the weights in air and in water (*i.e.* W — P) is the weight of water displaced. Since the volumes displaced in both cases are equal, the ratio of their weights is the specific gravity of the liquid, *i.e.*

$$\text{specific gravity of liquid} = \frac{\text{weight of liquid}}{\text{weight of equal volume of water}}$$
$$= \frac{W - Q}{W - P}.$$

Example.—*To find the specific gravities of glass and glycerine, from the following data:—*

Weight of a piece of glass in air		= 10 *grm.*
,, ,, ,,	*water*	= 6 *grm.*
,, ,, ,,	*glycerine*	= 5 *grm.*

Here weight of water displaced by glass = 10 — 6 = 4 grm.,

,, glycerine ,, ,, = 10 — 5 = 5 grm.,

and weight of glass = 10 grm.;

$$\therefore \text{ specific gravity of glycerine} = \frac{\text{weight of glycerine displaced}}{\text{weight of water displaced}}$$
$$= \tfrac{5}{4} = 1{\cdot}25;$$

and $\text{specific gravity of glass} = \tfrac{10}{4} = 2{\cdot}5.$

(*d*) *To find the specific gravity of a solid which is soluble in water.*

In Articles (*a*) and (*b*) of this section we assumed that the solid was insoluble in water. If, however, we require to use the Hydrostatic Balance to find the specific gravity of a solid which *is* soluble in water, we must first find a liquid of known specific gravity (*s*) in which it is insoluble and then proceed as follows:—

(i) weigh the solid in air (W),

(ii) weigh the solid in the liquid (Q).

Then the apparent loss of weight in the liquid is equal to the weight of the liquid displaced by the solid

Thus,

$$\text{weight of liquid displaced by solid} = W - Q,$$

$$\therefore \quad \text{weight of equal volume of water} = (W - Q)/s,$$

$$\text{specific gravity of solid} = \frac{\text{weight of solid}}{\text{weight of equal volume of water}}$$

$$= \frac{W}{(W-Q)/s}$$

$$= \frac{Ws}{W-Q}$$

Example—*To find the specific gravity of a substance soluble in water, but not in turpentine, from the following data*—

Weight of solid in air $= 32$ *grm*

" *turpentine* $= 3$ *grm*

specific gravity of turpentine $= 0.87$

The weight of turpentine displaced $= 32 - 3 = 29$ grm

But weight of turpentine $= 0.87$ (weight of equal volume of water),

$$\therefore \quad \text{weight of equal volume of water} = \frac{29}{0.87} = \frac{29 \times 100}{87} = \frac{100}{3} \text{ grm},$$

$$\text{specific gravity of solid} = \frac{32 \times 3}{100} = 0.96$$

The above method may also be used to find the specific gravity of a solid which is lighter than water by weighing it in a liquid of still smaller specific gravity, thus dispensing with the use of a sinker

64 Correction for displaced air when weighing

In finding specific gravities of solids, we supposed their weights found by weighing them in air with a common balance If great accuracy is required, it will be necessary either to weigh the bodies *in vacuo* or to allow for the fact that the bodies, as well as the set of weights employed, all displace more or less air, and therefore the apparent weight of a body in air is less than its true weight by the weight of this displaced air But the density of air is very small compared with that of most solids and liquids, being 0.0013 of that of water Hence the weight of the displaced air is in most cases so small a fraction of the weight of the body that no serious error is introduced by neglecting it altogether (see § 57)

It is easy, however, to make allowance for the displaced air, if necessary For when a body is placed in one pan of a pair of scales, and balanced by weights in the other, the apparent weights or resultant

forces tending to draw the body and weights towards the ground are equal. Hence

true weight of body − weight of air displaced by body
= weight of weights − weight of air displaced by weights . . . (i)

Let W be the true weight of the body and d its density and suppose that its apparent weight by the balance (*i.e.* the sum of the "weights") is W_0, and that the metal comprising these weights is of density d_0. Then if ρ is the density of the air, we have

$$\text{volume of body} = \frac{W}{d};$$

$$\therefore \quad \text{weight of air displaced by body} = \frac{W}{d}.\rho.$$

Again, $$\text{volume of "weights"} = \frac{W_0}{d_0};$$

$$\therefore \quad \text{weight of air displaced by weights} = \frac{W_0}{d_0}.\rho;$$

$\therefore$ Equation (i) becomes $$W - \frac{W\rho}{d} = W_0 - \frac{W_0}{d_0}\rho,$$

$$\text{i.e.} \quad W\left(1 - \frac{\rho}{d}\right) = W_0\left(1 - \frac{\rho}{d_0}\right);$$

$$\therefore \quad W = W_0\left(1 - \frac{\rho}{d_0}\right)\Big/\left(1 - \frac{\rho}{d}\right).$$

Thus the true weight (W) is found by multiplying the apparent weight (W_0) by

$$\frac{1 - \dfrac{\rho}{d_0}}{1 - \dfrac{\rho}{d}}.$$

Example.—*A piece of cork (specific gravity 0·24) apparently weighs ½ lb. when weighed by means of brass weights (specific gravity brass = 8·4). Find the true weight of the piece of cork if the specific gravity of air is* 0·0013.

From the above

$$W = W_0\left(\frac{1 - \dfrac{\rho}{d_0}}{1 - \dfrac{\rho}{d}}\right)$$

$$= 0{\cdot}5\left(1 - \frac{0{\cdot}0013 \times 62\frac{1}{2}}{8{\cdot}4 \times 62\frac{1}{2}}\right)\Big/\left(1 - \frac{0{\cdot}0013 \times 62\frac{1}{2}}{0{\cdot}24 \times 62\frac{1}{2}}\right)$$

$$= 0{\cdot}50265 \text{ lb. on simplification.}$$

65 Hydrometers

A hydrometer is an instrument for finding the specific gravity of liquids It is floated in the liquid and the principle on which it works is that the weight of a floating body is equal to the weight of liquid which it displaces There are two principal types of hydrometer —

(i) The *common hydrometer*, which determines the specific gravity of the liquid in which it is placed by the depth to which it sinks, the weight of the hydrometer being constant, and

(ii) *Nicholson's hydrometer*, which has a scale pan attached and weights are placed in this until a definite volume of the hydrometer is immersed

We proceed to consider these two hydrometers in more detail

(a) *The common hydrometer*

This hydrometer consists of a glass tube or stem blown out into two bulbs (Fig 121) The stem and the upper bulb are filled with air, the lower bulb being filled with mercury so that when the hydrometer is in the liquid it floats upright with the whole of the bulb and part of the stem immersed The stem is graduated the marks at different levels giving the specific gravities of the liquids in which the hydrometer would sink to these levels

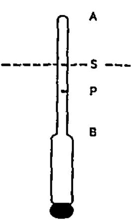


Fig 121

Let the stem AB, supposed uniform, be of cross section a Then if the hydrometer floats in water with the point S in the surface, there is a volume a AS out of the water Consequently if the whole volume of the hydrometer is V then the volume of water displaced when floating is $V - a$ AS

If water weighs w per unit of volume the weight of water displaced is therefore $w(V - a\text{ AS})$,

and this must equal the weight of the instrument

If the hydrometer is now floated in a liquid of specific gravity s (> 1), the volume now immersed will be less suppose the point P is now in the surface In this case, the volume immersed is

$$V - a\text{ AP},$$

so that the weight of liquid now displaced is

$$ws(V - a\text{ AP})$$

This must also be equal to the weight of the instrument, hence

$$ws(V - a\text{ AP}) = w(V - a\text{ AS}), \qquad s = \frac{V - a\text{ AS}}{V - a\text{ AP}}$$

This enables the hydrometer to be graduated before it is sold.

Example 1.—*To find the specific gravity of a liquid, having given that a hydrometer weighing* $1\frac{1}{2}$ *oz. sinks in it until* 2·4 *cub. in. are immersed.*

Here weight of 2·4 cub. in. of liquid = 1·5 oz.;

$$\therefore \quad \text{weight of 1 cub. in. of liquid} = \frac{1\cdot5}{2\cdot4} \text{ oz.} = \tfrac{5}{8} \text{ oz.},$$

and weight of 1 cub. ft. of liquid $= 1728 \times \frac{5}{8}$ oz. $= 1080$ oz.

But weight of 1 cub. ft. of water = 1000 oz.;

$$\therefore \quad \text{specific gravity of liquid} = \tfrac{1080}{1000} = 1\cdot08.$$

Example 2.—*To find the density of a liquid in which a common hydrometer floats with* $3\frac{1}{2}$ *in. of its stem immersed, having given that the diameter of the stem is* 0·2 *in., the volume of the two bulbs is* 0·754 *cub. in., and the weight of the hydrometer* $\frac{1}{2}$ *oz.*

Here the portion of the stem immersed is a cylinder of height $3\frac{1}{2}$ in., the radius of whose base is

$$\tfrac{1}{2} \times 0\cdot2 = 0\cdot1 \text{ in.}$$

Hence the volume of the immersed portion of the stem

$$= \tfrac{22}{7} \times (0\cdot1)^2 \times \tfrac{7}{2} = 0\cdot11 \text{ cub. in.}$$

Moreover, the volume of the bulbs = 0·754 cub. in.

Hence the whole volume of the displaced liquid

$$= 0\cdot754 + 0\cdot11 = 0\cdot864 \text{ cub. in.}$$

But weight of displaced liquid = weight of hydrometer = $\frac{1}{2}$ oz.;

$$\therefore \quad 0\cdot864 \text{ cub. in. of liquid weighs } \tfrac{1}{2} \text{ oz.};$$

$$\therefore \quad 1 \text{ cub. in. of liquid weighs } \frac{1}{2 \times \cdot864} = \frac{1}{1\cdot728} \text{ oz.};$$

$$\therefore \quad 1 \text{ cub. ft. of liquid weighs } \frac{1728}{1\cdot728} \text{ oz.} = 1000 \text{ oz.}$$

Hence the liquid is of the same density as water, and its specific gravity is unity.

Example 3.—*The stem of a hydrometer is divided into* 100 *equal parts. It reads* 0 *in water and* 100 *in liquid of specific gravity* 0·8. *To find the specific gravity for which the hydrometer reads* 50.

Let O, Q (Fig. 122) be the points marked 0, 100; P the point marked 50.

Let V be the volume of water whose weight is equal to that of the hydrometer. Then V is the volume of water displaced when the hydrometer floats in water.

$$\therefore \quad \text{volume displaced by portion below O} = \text{V}.$$

When the hydrometer floats in the lighter liquid of specific gravity 0·8, it displaces an equal weight, and therefore a greater volume, of liquid

$$\text{volume displaced by portion below } Q = V \div \cdot 8 = 1{\cdot}25V,$$

$$\therefore \text{ volume of stem } OQ = 1{\cdot}25V - V = 0{\cdot}25V,$$

that is, $\quad$ volume of 100 divisions of stem $= 0{\cdot}25V$,

$$\therefore \text{ volume of 50 divisions of stem} = 0{\cdot}125V,$$

$$\therefore \text{ volume displaced by portion below } P = V + 0{\cdot}125V = 1{\cdot}125V$$

This is the volume displaced by the hydrometer in the given liquid, and its weight is equal to the weight of water displaced

$$\therefore \text{ wt of vol } 1{\cdot}125V \text{ of given liquid} = \text{wt of vol } V \text{ of water},$$

$$\text{wt of vol 1 of given liquid} = \text{wt of vol } 1 \div 1{\cdot}125 \text{ of water},$$

$$\therefore \text{ required specific gravity of liquid} = 1 \div 1{\cdot}125 = \tfrac{8}{9} = 0{\cdot}\dot{8}$$

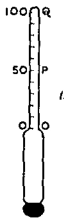


Fig 122

[NOTE —Although the mark 50 is midway between the marks 0 and 100, the required specific gravity is NOT midway between the corresponding specific gravities, for its value is $0{\cdot}\dot{8}$, and *not* 0·9 as might on first thoughts be expected]

Example 4 —*With the data of the last example to find the specific gravity of a liquid whose reading is 28*

We have seen that

$$\text{vol of 100 divisions of stem} = 0{\cdot}25V,$$

$$\text{vol of 28 divisions} = \tfrac{28}{100} \times 0{\cdot}25V = 0{\cdot}07V$$

$$\text{vol displaced by hydrometer in given liquid} = 1{\cdot}07V$$

$$\text{wt of vol } 1{\cdot}07V \text{ of liquid} = \text{wt of vol } V \text{ of water}$$

$$\text{specific gravity of liquid} = 1 \div 1{\cdot}07 = 0{\cdot}9346, \text{ nearly}$$

Example 5 —*A hydrometer consists of a bulb surmounted by a rod of uniform cross section When immersed in liquids of specific gravities s_1, s_2, lengths a and b respectively of the rod are above the liquids Prove that when immersed in a mixture of equal weights of the liquids a length $\frac{1}{2}(a+b)$ of the rod will be above the mixture* (*Inter Eng*)

Let W be the weight of the hydrometer and V its total volume Then if the cross sectional area of the rod is d, the weight of liquid displaced when a length a of the rod is above the liquid is

$$(V - ad)\, s_1 w,$$

and this equals W, the weight of the hydrometer

Similarly $\qquad W = (V - bd)\, s_2 w$

If the specific gravity of the mixture is S, and when the hydrometer is floated in this mixture there is a length h of the rod above the surface,

then also $$W = (V - kd)\,Sw.$$
In each case, w is the weight per unit volume of water.

Thus we have a series of relations from the constancy of the weight of the hydrometer:—

$$W = (V - ad)\,s_1 w = (V - bd)\,s_2 w = (V - kd)\,Sw \quad \ldots\ldots \text{(i)}$$

We may obtain the specific gravity of the mixture (S) in terms of the specific gravities of the two constituent liquids (s_1 and s_2). If we suppose a volume v_1 of the liquid of specific gravity s_1 is mixed with a volume v_2 of the liquid of specific gravity s_2, then since the weights are equal

$$v_1 s_1 = v_2 s_2 \quad \ldots\ldots \text{(ii)}$$

and $$(v_1 + v_2)\,S = v_1 s_1 + v_2 s_2 \quad \ldots\ldots \text{(iii)}$$

Eliminating v_1 and v_2 from (ii) and (iii),

$$\left(v_1 + \frac{v_1 s_1}{s_2}\right) S = 2 v_1 s_1;$$

$$\therefore \quad S = \frac{2 s_1 s_2}{s_1 + s_2}$$

$$= \frac{2}{\dfrac{1}{s_1} + \dfrac{1}{s_2}} \quad \ldots\ldots \text{(iv)}$$

The relations (i) may be expressed

$$V - ad = \frac{W}{w} \cdot \frac{1}{s_1} \quad \ldots\ldots \text{(v)}$$

$$V - bd = \frac{W}{w} \cdot \frac{1}{s_2} \quad \ldots\ldots \text{(vi)}$$

$$V - kd = \frac{W}{w} \cdot \frac{1}{S} \quad \ldots\ldots \text{(vii)}$$

Adding (v) and (vi),

$$2V - (a + b)\,d = \frac{W}{w}\left(\frac{1}{s_1} + \frac{1}{s_2}\right)$$

$$= \frac{W}{w} \cdot \frac{2}{S} \text{ from (iv)}$$

$$= 2\,(V - kd) \text{ from (vii)}$$

$$= 2V - 2kd;$$

$$\therefore \quad -(a + b)\,d = -2kd;$$

$$\therefore \quad k = \tfrac{1}{2}(a + b).$$

(*b*) *Nicholson's hydrometer.*

This is a constant volume hydrometer, since, when it is floated in a liquid, weights are added until a definite volume is immersed.

It consists of a hollow metal container A (Fig 123) to the base of which is attached a fixed heavy metal cup B Above the container projects a metal stem supporting a scale pan C, and on this stem is a fixed mark D In whatever liquid the hydrometer is floated, weights are placed in the scale pan until D is just in the surface Then in each case, exactly the same volume is immersed

To find the specific gravity of a liquid The following observations have to be made

(i) Find the weight (W) of the hydrometer in air

(ii) Float the instrument and add weights to the scale pan until the mark D is in the surface Let these weights be P

(iii) Now float the hydrometer in the liquid whose specific gravity is required and suppose that it requires a weight Q to sink the mark D to the surface level

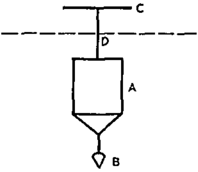


Fig 123

From the first two of these measurements we know the weight of water displaced by that part of the hydrometer below the fixed mark D because this weight of water displaced is equal to the total weight of the hydrometer, together with the weights in the scale pan, *i e* W + P

From the third determination we have that the weights of the *same volume* of liquid displaced is W + Q

$$\text{Hence} \quad \text{specific gravity of liquid} = \frac{\text{weight of liquid displaced}}{\text{weight of equal volume of water}}$$

$$= \frac{W + Q}{W + P}$$

Example —*To find the specific gravity of brandy by means of a Nicholson s hydrometer weighing* 60 *grm*, *having given that* 23 7 *grm are required in the upper scale pan to sink the hydrometer to the fixed mark when placed in brandy, and that* 40 *grm are required to sink it to the same mark in water*

At first observation, the total weight supported by the brandy

$$= 60 + 23\ 7 = 83\ 7 \text{ grm}$$

Hence weight of brandy displaced = 83 7 grm

At the second observation, we have, in like manner,

weight of water displaced = 60 + 40 = 100 grm

But the volumes of the brandy and water displaced are equal;

$$\therefore \quad \text{specific gravity of brandy} = \frac{83{\cdot}7}{100} = 0{\cdot}837.$$

To find the specific gravity of a solid. Nicholson's hydrometer may be used to determine the specific gravity of a solid, provided the solid is small enough to be placed in the metal cup B. The following measurements need to be made:—

(i) The hydrometer alone is floated in water and weights are added to the upper scale-pan until the mark D is in the surface. Let these weights be P.

(ii) The solid is now placed in the upper scale-pan, together with the weights (Q) now necessary to sink the mark D to the surface. Hence the weight of the solid is P − Q.

(iii) The solid is now placed in the *lower* metal cup B, the whole immersed in water and the weights (R) which must be placed in the upper scale-pan to sink the mark D to the surface, is noted.

The difference between (ii) and (iii) is that in (iii) there is the additional upthrust equal to the weight of water displaced by the solid. Consequently greater force is needed to depress the hydrometer to the mark D in (iii) than in (ii), *i.e.* $R > Q$ and

$$\text{weight of water displaced by solid} = R - Q.$$

Hence

$$\text{specific gravity of solid} = \frac{\text{weight of solid}}{\text{weight of equal volume of water}} = \frac{P - Q}{R - Q}.$$

Example.—*A Nicholson's hydrometer when placed in water required a weight of* 40 *grm. in the upper scale to sink it to the fixed mark. When a piece of silver was placed in the upper scale-pan,* 8·5 *grm. was required to sink it, and when the silver was placed in the lower scale-pan,* 11·5 *grm. was required in the upper. To find the specific gravity of silver.*

Here, when the silver was placed in the upper scale-pan, 40 − 8·5 or 31·5 grm. had to be taken out in order to make the total weight the same as before.

Therefore the weight of the silver was 31·5 grm.

When the silver was transferred to the lower pan and immersed, we had to add 11·5 − 8·5 or 3 grm. to the upper pan to counteract the upward thrust of the water on the silver;

$$\therefore \quad \text{weight of water displaced by silver} = 3 \text{ grm.}$$

$$\therefore \quad \text{specific gravity of silver} = \frac{31{\cdot}5}{3} = 10{\cdot}5.$$

66. U-Tubes

The specific gravities of two liquids which do not mix may be compared very simply by means of a U tube

Let the two liquids have specific gravities s_1 and s_2 and suppose a quantity of each is poured into the U tube as indicated in Fig 124 If A represents the common surface and B a point in the same horizontal plane in the other limb of the U tube, then the pressure intensity at A equals the pressure intensity at B (see Chapter III), and if there is a length l_1 of the liquid of specific gravity s_1 then

$$\text{pressure intensity at A} = P + l_1 s_1 \times 62\tfrac{1}{2} \text{ lb wt /sq ft}$$

where P is the atmospheric pressure

If there is a length l_2 of the liquid of specific gravity s_2 above B, then

$$\text{pressure intensity at B} = P + l_2 s_2 \times 62\tfrac{1}{2} \text{ lb wt /sq ft}$$

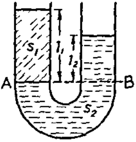


Fig 124

Equating these two pressure intensities

$$l_1 s_1 = l_2 s_2$$

$$\text{or} \quad \frac{l_1}{l_2} = \frac{s_2}{s_1},$$

i e the heights of the liquids above the common surface are inversely proportional to their specific gravities Thus, if one of the liquids is water, or any liquid of known specific gravity, the specific gravity of the other liquid is obtained directly from the relation above, having found the lengths l_1 and l_2

An example involving this principle has already been worked out in Chapter III (Example 1, § 22)

67. Hare s hydrometer

This hydrometer is, in effect, an inverted U tube, but it has the advantage that it may be used to compare the specific gravities of two liquids without them coming into contact with each other, so that it does not matter whether they would mix or not

The hydrometer consists of two hollow vertical tubes (A and B) connected by a hollow horizontal tube, C, from which part of the air enclosed may be drawn off (Fig 125)

The two vertical tubes have their ends placed in the liquids whose specific gravities (s_1 and s_2) are to be compared Some air is then drawn off from the outlet D by a pump so that the air in the hydrometer is at less than atmospheric pressure and it is then sealed off by closing the stopper or valve E This reduction in pressure inside the hydrometer causes the liquids to rise in the vertical tubes Suppose the heights above the surfaces F and G are l_1 and l_2 respectively If P

represents the atmospheric pressure and P′ the pressure to which the air in the hydrometer is reduced, then the pressure intensity at $F = P$. But the pressure intensity at F equals the pressure intensity at a point in the left hand tube at the same level,

i.e. $$P = P' + l_1 s_1 w \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where w = weight per unit volume of the standard substance.

Again, the pressure intensity at G (which is at atmospheric pressure P) is equal to the pressure intensity at a point at the same level as G, in the right hand tube and this pressure intensity is $P' + l_2 s_2 w$. Thus

$$P = P' + l_2 s_2 w \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Equating (i) and (ii),

$$l_1 s_1 = l_2 s_2,$$

giving the same relationship as for the U-tube.

It should be noticed that there is no need for F and G to be at the same level. Wherever they are, these surfaces will be at atmospheric pressure, which is all that the above proof requires.

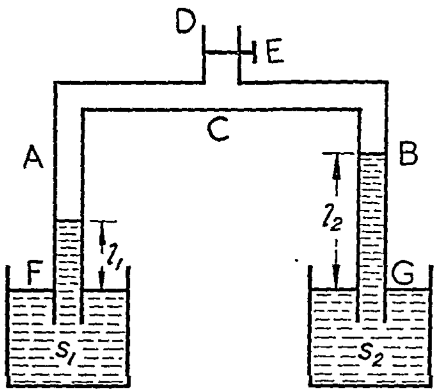


Fig. 125.

68. Statistical examination of numerical results

Consider again the table of specific gravities given at the beginning of this chapter. It will be noticed that all values are given to two places of decimals, but that in the case of glass and oak a range of variation is given. In fact there will be slight variations in many of the others. For instance, since all the woods will have some moisture content, which may vary, the specific gravities for different specimens may easily have different figures in the second place of decimals.

Now suppose that an experiment is performed in a laboratory to determine the specific gravity of, say, gold. The specific gravity of gold, from our table, is given as 19·25, and if a value obtained from an experiment was 19·17 this would be considered quite a good result. If, on the other hand, the value obtained was, say, 16·75, it would be assumed that something had gone wrong with the experiment, and it would be necessary to check that the sample was pure gold and that the method had been correctly carried through. But assuming that

all possible care is taken, it still will not in general be found that the value of the specific gravity obtained is exactly 19.25. It may be slightly more or slightly less. Moreover, if the experiment is repeated with the same sample it is probable that a result differing from the first will be obtained. This is because, in spite of all possible precautions, a large number of tiny errors creep in which, although they may be individually minute, nevertheless together affect the final value obtained. It is these accidental errors, as they are termed, which cause the slight deviation from 19.25 and in general they will sometimes give a result in excess of 19.25 and sometimes less than it. Thus the "best" value for the specific gravity will be found by repeating the experiment several times and taking the *average* or *mean* of the results.

The more elaborate the precautions taken in performing the experiments, the closer will be the values to 19.25, in other words the values will be more concentrated about the true value.

When dealing with a large number of measurements of the same quality, in addition to the determination of the mean, it is frequently useful to have a measure of this concentration, or degree of scattering, of the values about the mean. The most convenient measure of this is called the *standard deviation* and for fuller information on this and allied topics, a book on statistics should be consulted.

Exercises IX

(1) The weight of a specific gravity bottle when empty is 42 grm., and when full of water and glycerine respectively, its weight is 222 grm. and 269 grm. Find the specific gravity of glycerine.

(2) A piece of chalk weighs 48 grm. in air and 28 grm. in water. Find its specific gravity.

(3) A specific gravity bottle full of water weighs 44 grm., and when some pieces of iron weighing 10 grm. in air are introduced into the bottle and the bottle again filled up with water, the combined weight is 52.7 grm. What is the specific gravity of the iron?

(4) 100 grm. of a certain powder are placed in a specific gravity bottle weighing 50 grm. and capable of holding 500 grm. of water. The bottle is filled with ether of specific gravity 0.72, and the whole is then found to weigh 474 grm. Find the specific gravity of the substance forming the powder.

(5) A piece of iron (specific gravity = 7.21) weighing 216.3 grm. is attached to a piece of cork weighing 36 grm., and the weight of both in water is 36.3 grm.; find the specific gravity of the cork.

(6) A ball of metal weighs 9 lb. in air and 8 lb. when suspended in water. What would be the specific gravity of a liquid in which it would weigh $7\frac{1}{4}$ lb.?

(7) An iron shell is found to lose half its weight when weighed in water. What portion of its volume is hollow? (Specific gravity of iron = 7.2.)

(8) A body which is soluble in water weighs 27 grm., and when weighed in oil of specific gravity 0.9 its weight is $20\frac{1}{4}$ grm. Find its specific gravity.

(9) A piece of silver and a piece of gold are suspended from the arms of an equal armed balance beam, which is in equilibrium when the silver is immersed in alcohol (density 0.85) and the gold in nitric acid (density 1.5). The densities of the silver and gold being 10.5 and 19.3 respectively, what are their relative masses?

(10) Prove that a brass pound weight made to balance, on scales, the standard pound of platinum in air, is too great by the fraction

$$\left(\frac{A}{B}-\frac{A}{P}\right)\bigg/\left(1-\frac{A}{B}\right),$$

where A, B, P denote the specific gravity of air, brass and platinum respectively.

(11) When the common hydrometer floats in water $\frac{9}{10}$ of its volume is immersed, and when it floats in milk $\frac{90}{103}$ of its volume is immersed. Find the specific gravity of milk.

(12) Prove that the bulb of a common hydrometer must have $10\frac{1}{3}$ times the volume of the stem for reading specific gravity between 1.24 and 1.36.

(13) A Nicholson's hydrometer weighs $3\frac{3}{4}$ oz., and requires a weight of $1\frac{3}{4}$ oz. to sink it to the fixed mark in water. What weight will be required to sink it to the fixed mark in a liquid whose density is 2.5?

(14) A piece of marble weighing 142 grm. is placed in the upper dish of a Nicholson's hydrometer, and it is found that an additional weight of 40 grm. is required to sink the hydrometer to a fixed mark in its stem. When the marble is placed in the lower dish, it is found that 90 grm. are necessary. What is the specific gravity of the marble?

(15) A U tube is fixed with both limbs vertical and contains mercury (sp. gr. 13.6). When a quantity of a certain liquid A is poured into the left hand limb of the tube and the liquids are allowed to come to rest, the whole of the liquid A is in the left hand limb and the free surfaces of A and of the mercury are at heights of 35.3 and 5.4 cm. respectively above the surface of separation of the two liquids. Find the specific gravity of liquid A.

Further, find the height of a column of a third liquid B (sp. gr. 4.7) that must be introduced into the right hand limb in order that the mercury may stand at the same level in each limb. (H.S.C., I.)

(16) A piece of an alloy weighing 96 grm. is composed of two metals, the specific gravity of one being 11.4 and of the other 7.4. If the weight of the alloy in water is 86 grm., find the weight of each metal in the alloy. (Inter. Sc.)

(17) A body when weighed in water has an apparent weight of 3 lb. and when weighed in a liquid of specific gravity 0.8 has an apparent weight of

3 1 lb Find the weight of the body and its apparent weight when weighed in a liquid of specific gravity 0·6

If 1 cub ft of water weighs 62 5 lb and the body is a cube, find the length of its edge (H.S C., I)

(18) A light rigid rod ACB is balanced on a knife edge at C Two unequal masses P and Q, of densities ρ_1 and ρ_2, are hung by fine wires attached to the rod, at distances x_1 and y_1 from C, so that the rod remains horizontal When P and Q are totally immersed in liquids of densities σ_1 and σ_2, the balance of the rod is restored *either* by moving Q to a distance y_2 from C, keeping x_1 constant, *or* by keeping y_1 constant and moving P to a distance x_2 from C Prove

(i) $$x_1y_2 = x_2y_1,$$

(ii) $$\left\{1 - \frac{\sigma_1}{\rho_1}\right\}^2 \Big/ \left\{1 - \frac{\sigma_2}{\rho_2}\right\}^2 = x_1y_2/x_2y_1$$ (H S C , I)

(19) The two branches of a U tube are close together One branch contains mercury and the other water so that their common surface is at the lowest point the difference in height of their upper surfaces is d If $1/n$th of the water is taken out and poured into the other branch above the mercury, show that in the new equilibrium position the difference in heights of the upper surfaces is $(n - 2)\,d/n$.

ANSWERS

1. 1 26	2. 2 4	3 7 692	4 2
5 0 1935	6 1 5	7 13/18	8 3 6
9 1 00352 1	11 1·03	13 10 oz wt	14 2 84
15 2 08, 15 6 cm		16 6? 7 grm , 33 3 grm	
17 3 5 lb wt , 3 2 lb wt , 2 4 in			

CHAPTER X

GASES

69. Introduction

In the preceding chapters we have proved many theorems concerning fluids in general, but the application of them has mainly been concerned with liquids. This chapter is devoted to gases.

Let us consider first of all what we know of gases so far. In § 17, we stated that the pressure intensity at any depth of a liquid depends on the atmospheric pressure at the surface; in other words it depends on the weight of air pressing on the surface of the liquid. This atmospheric pressure we shall consider in more detail later on in this chapter.

Again, in § 52 we proved Archimedes' Principle true for all fluids, gases as well as liquids, and in consequence of this we found in § 57 the correction for the displaced air when a body floated partly immersed in a liquid and partly in a gas. This, in turn, was utilised in § 64 in the determination of the correction for the displaced air to be applied when weighing a solid in air as distinct from *in vacuo*. Here we took the specific gravity of air as 0·0013, *i.e.* taking one cubic foot of water as weighing 62·5 lb. or 1000 oz.,

1 cub. ft. of air weighs $0{\cdot}0013 \times 1000$ oz. $= 1{\cdot}3$ oz.

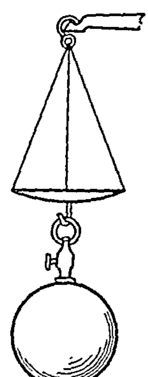


Fig. 126.

70. To find the density of air

Although we have stated the density of air, we have not yet described an experiment from which this may be determined. We proceed to this now:—

A large glass flask fitted with a tap or stopcock is taken; the air in it is completely exhausted by means of an air-pump, and the tap is closed. The flask is then weighed with a balance (Fig. 126). On opening the tap, air rushes into the flask and depresses the scale-pan carrying it; hence the flask is heavier than before. The difference in weight is found by again weighing, and is evidently equal to the weight of air which entered the flask; and, if the volume of the flask be determined, the density and specific gravity of the air may be found.

Example.—*A flask weighs* 273·4 *grm. when empty,* 276·5 *grm. when filled with air, and* 2658·4 *grm. when filled with water. To find the weight of a litre of air.*

Weight of air in flask = 276 5 — 273 4 = 3 1 grm

weight of equal volume of water = 2658 4 — 273 4 = 2385 grm ,

specific gravity of air = 3 1 — 2385 = 0 0013,

but a litre of water weighs 1000 grm ,

weight of a litre of air = 1 3 grm

It is interesting to consider one of the earliest ways of showing that air exerts a pressure In the seventeenth century, Otto von Guericke of Magdeburg designed two hollow hemispheres which fitted together with an air tight join When the air was withdrawn from these tightly fitting hemispheres by means of an air pump, it was found that considerable force was needed to pull them apart This force is required to overcome the resultant thrusts produced by the atmosphere on the outer surfaces of the hemispheres We have stated elsewhere that the pressure of the atmosphere is approximately 14 7 lb wt /sq in hence if we know the diameter of the sphere we may determine the exact force required Suppose the hemispheres have a diameter of one foot, then from § 50 the resultant thrust of the air on the curved surface of the hemisphere

$$= \pi (6)^2 \times 14\ 7 \text{ lb wt}$$
$$= 1663\ 2 \text{ lb wt}$$

Therefore at least this force must be applied to each hemisphere in order to pull them apart

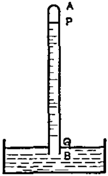


Fig 127

71 Measurement of atmospheric pressure

It was in 1643 that an experiment was devised by Torricelli to measure the pressure of the atmosphere and this experiment resulted in the invention of the barometer

To perform the experiment or to construct a barometer in its simplest form a glass tube about 33 in long and closed at one end is completely filled with mercury The open end is then closed with the finger the tube inverted into a cup of mercury, and the finger then removed care being taken not to allow any air to get into the tube The mercury will at once sink and leave a clear space at the top of the tube and the height of the column of mercury above the surface in the cup will be found to be about 30 inches or 76 cm

Suppose AB (Fig 127 represents the tube and P the level of the mercury We proved in § 15 that the pressure intensity is the same at all points at the same level in a liquid Hence the pressure intensity in the tube at Q is equal to the pressure intensity in the surface of the surrounding mercury, i e the pressure intensity in the tube at Q is equal to the atmospheric pressure

Now the pressure intensity in the tube at Q is due to the length of mercury PQ, together with the pressure intensity at P. But the pressure intensity at P is zero, since AP is practically a vacuum. Hence the length PQ measures the atmospheric pressure. In fact, AP contains a minute quantity of mercury vapour, but this provides only an insignificant pressure intensity at P—we may regard it as zero.

Suppose, for clarity, the length PQ is 30 in. and we wish to determine the atmospheric pressure.

The pressure of the atmosphere per square inch

= weight of a column of mercury, 30 in. high, on a base of 1 sq. in.

= weight of 30 cub. in. of mercury

= $13{\cdot}6 \times$ weight of 30 cub. in. of water

(taking specific gravity of mercury as 13·6)

$$= 13{\cdot}6 \times 30 \times \frac{62{\cdot}5}{12^3} \text{ lb. wt.}$$

= 14·75 lb. wt.,

i.e. the atmospheric pressure corresponding to 30 in. of mercury is 14·75 . . . lb. wt. per sq. in.

In fact, the length PQ is continuously varying, indicating that the atmospheric pressure is changing slightly all the time, and this change may be used to predict the weather, since experience has shown that certain changes of weather are generally accompanied by certain changes of atmospheric pressure. For instance, rainy weather is usually preceded by a decrease in the pressure of the atmosphere, and an improvement in the weather generally coincides with an increase in the atmospheric pressure.

72. Barometers

Any instrument designed for measuring the atmospheric pressure is called a barometer. In its simplest form it consists of a tube containing mercury as described in § 71, with a scale attached for reading the height of the mercury column. Mercury is suitable in the instrument because of its high specific gravity, and the barometer need only be about 34 in. long. If, on the other hand, other liquids were used, the barometers would be much more unwieldy, although, also, much more sensitive. If water were used, the length of water column corresponding to 30 in. of mercury would be $30 \times 13{\cdot}6$ in., *i.e.* 34 ft.

Another objection to the water barometer is the difficulty of retaining a vacuum at the top of the tube, since water evaporates freely into the vacant space. Glycerine has not this disadvantage, and since its specific gravity is 1·26, the glycerine barometer is more than ten times as sensitive as a mercury barometer.

The most convenient form for a mercury barometer is that of a U tube with unequal arms and with unequal diameters (Fig 128)

The mercury in the short limb is open to the atmosphere and the difference in height (PQ) gives the length of the mercury column whose weight measures the atmospheric pressure This type is known as the *Siphon Barometer*

Another useful form of barometer is known as the *Aneroid Barometer* This is a hollow metal box exhausted or nearly exhausted of air The atmospheric pressure tends to force in the top of the box, but is resisted by the elasticity of the metal, which acts like a spring When the pressure increases or decreases, the lid sinks or rises slightly, and moves a pointer which indicates the pressure on a dial This dial is graduated in " inches " or ' millimetres," corresponding to the readings of a mercurial barometer The aneroid is chiefly used on account of its portability

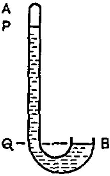


Fig 128

73 Boyle's Law

This experimental law, giving the relation between the volume and pressure of a gas at constant temperature, was discovered by Robert Boyle in 1662 and, independently, by Mariotte in France in 1679 It is as follows —

If the temperature of a given mass of gas is kept constant, the pressure varies inversely as the volume, i e if p, v denote the pressure and volume respectively,

then $$p \propto \frac{1}{v}$$

i e $$p = \frac{1}{v} \times \text{constant},$$

or $$pv = \text{constant}$$

Alternatively, if a mass of gas has a volume v_1 when the pressure is p_1, a volume v_2 when the pressure is p_2 a volume v_3 when the pressure is p_3 and so on, then

$$p_1 v_1 = p_2 v_2 = p_3 v_3 =$$

A method of verifying Boyle's Law experimentally is given in § 74

Example 1.—*A mass of air at atmospheric pressure occupies 44 cub in To find the pressure intensity when the volume is reduced to 24 cub in, taking an atmosphere as 15 lb per sq in*

Let p be the required pressure intensity in pounds per square inch Then, by Boyle's Law,

$$p \times 24 = 44 \times 15;$$

$$\therefore \quad p = \frac{44 \times 15}{24} = \frac{11 \times 5}{2} = 27\tfrac{1}{2} \text{ lb. per sq. in.}$$

Example 2.—*A bubble of air rises from the bottom of a lake, and its diameter has doubled when it reaches the surface. To find the depth of the lake.*

The volume of a sphere is proportional to the cube of its diameter [$\because$ volume $= \frac{4}{3}\pi r^3 = \frac{1}{6}\pi(\text{diameter})^3$];

$\therefore$ volume at surface = 8 times volume at bottom.

Therefore, by Boyle's Law,

pressure at surface = $\frac{1}{8}$ pressure at bottom;

$\therefore$ pressure at bottom = 8 atmospheres.

Now, taking the height of the water barometer as 34 ft., the pressure increases 1 atmosphere for every 34 ft. descended. But the difference of pressure at the top and bottom is $8 - 1$, or 7 atmospheres;

$\therefore$ required depth of lake $= 34 \times 7 = 238$ ft.

Example 3.—*A cylinder, with its axis vertical, is closed at the top by a sliding piston of weight* 6 *lb. and contains gas. When a weight of* 1 *lb. is placed on top of the piston, the piston sinks a distance of* 1 *in. Find how much farther it will sink when an additional* 1 *lb. weight is placed on it. (Neglect atmospheric pressure.)* *(Inter. Eng.)*

Let the length of the gas chamber be h in., and the cross-section s sq. in.

Let Fig. 129 represent the three cases. Originally Fig. 129 (a), the volume of gas is sh cub. in. and the total pressure is 6 lb. wt.

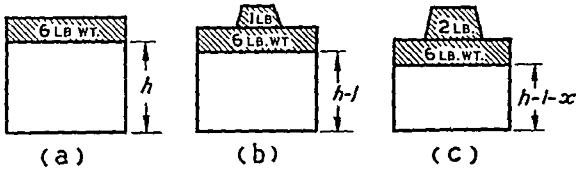


Fig. 129.

Thus, originally,

pressure × volume $= 6sh$ .... (i)

In the second case, Fig. 129 (b), when an additional 1 lb. wt. is added, the total pressure is 7 lb. wt. and the volume is $s(h - 1)$ cub. in.

Hence pressure × volume $= 7s(h - 1)$ .... (ii)

In the third case, Fig. 129 (c), the total pressure is 8 lb. wt. and the total volume is now only $s(h - 1 - x)$ cub. in., where x in. is the distance further depressed.

and since the pressure intensity at B is atmospheric,

$$30 = 30 - t + \frac{15}{2+t},$$

$$t = \frac{15}{2+t},$$

$$t^2 + 2t - 15 = 0,$$

$$(t+5)(t-3) = 0,$$

$t = 3$ is the only possible solution,

New height of barometer $= 30 - 3$ in

$= 27$ in

When the reading of the faulty barometer is x in, Fig 131 (c), the volume of the air is $(32 - x)s$ cub in, and if the pressure is p, we have as before

$$ps(32 - x) = \tfrac{1}{2}s\,30,$$

i e
$$p = \frac{15}{32 - x}$$

Then true height of barometer

$=$ pressure intensity at Q

$= x +$ pressure intensity at P

$= x + p$

$= x + \dfrac{15}{32 - x}$ in

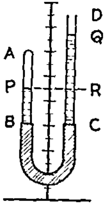


Fig 132

74 Experimental verification of Boyle's Law

Boyle's Law may be verified experimentally as follows —

Two glass tubes AB CD (Fig 132) are connected by means of flexible rubber tubing and the whole fixed to a vertical stand fitted with a vertical scale The rubber tubing and the tubes contain mercury between P and Q and air is imprisoned in the upper part of AB The tube CD can slide vertically up and down, changing the pressure, and consequently the volume of the imprisoned air In any position, the volume of the air in AB will be proportional to the length AP and the pressure on this air is the pressure intensity at P If h is the height of the mercury barometer,

pressure intensity at P $=$ pressure intensity at R

$= h + \text{QR}$

It will be found experimentally that AP is inversely proportional to $h + \text{QR}$,

i e
$$\text{AP}(h + \text{QR}) = \text{constant},$$

i e
$$\text{volume} \times \text{pressure} = \text{constant}$$

To verify Boyle's Law for any other gas, it is only necessary to substitute that gas for air in the tube AB.

75. Relations between pressure, density, and temperature of a gas

Suppose a mass of gas has an original pressure p_0, original volume v_0, and original density ρ_0, and that at some subsequent stage by altering either the pressure or volume but *not* the temperature these values are p, v, ρ.

Then, from Boyle's Law,

$$p_0 v_0 = pv \quad \text{(i)}$$

Since the mass of gas must remain unaltered we also have

$$\rho_0 v_0 = \rho v \quad \text{(ii)}$$

Dividing (i) by (ii),

$$\frac{p_0}{\rho_0} = \frac{p}{\rho} \quad \text{(iii)}$$

Hence $\frac{p}{\rho}$ is constant for a given gas,

i.e.

$$p = c\rho \quad \text{(iv)}$$

where c is a constant.

In other words, the pressure in a given kind of gas at a given temperature is proportional to its density.

Now suppose that the temperature is altered but that the pressure remains constant, there will be a change in volume because an increase in temperature will cause the gas to expand. The relation between the volumes is known as *Charles' Law*, having been experimentally discovered in 1787 by J. Charles; it may be stated thus:—

If the pressure of a given mass of gas is kept constant, for each increase of 1° C. in its temperature, its volume increases by a definite fraction of its volume at 0° C.

Thus, if v be the volume of a given mass of gas at t° C., and v_0 the volume at 0° C., the increase in volume per degree Centigrade is αv_0 where α is a constant,

i.e. volume increase for t° C. $= \alpha v_0 t$;

$$\therefore \quad v = v_0 + \alpha t v_0$$
$$= v_0(1 + \alpha t) \quad \text{(v)}$$

If ρ_0, ρ be the densities at 0° C. and t° C. respectively we have from (ii), which is still true because the mass is unaltered.

$$\frac{\rho_0}{\rho} = \frac{v}{v_0}.$$

Hence (v) becomes

$$\rho_0 = \rho(1 + \alpha t) \quad \text{(vi)}$$

For air and most gases a is found experimentally to be approximately $\frac{1}{273}$. From equation (iv), we have that the density is proportional to the pressure if the *temperature* is constant,

i.e. $$\rho \propto p \text{ if } t \text{ constant} \qquad \text{(vii)}$$

Also, from (vi), the density $(\rho) = \rho_0/(1 + at)$ when the *pressure* is constant, and since ρ_0 is a constant for the gas, we may write

$$\rho \propto \frac{1}{1 + at} \text{ if } p \text{ constant} \qquad \text{(viii)}$$

We want to discover from these two relations (vii) and (viii), the connection between ρ, p, t when they are all liable to vary.

We write (vii) as

$$\rho = p \times \text{a function of } t$$
$$= p \times f(t), \text{ say} \qquad \text{(ix)}$$

for when t is constant $f(t)$ is constant and we have simply relation (vii).

Similarly (viii) may be written

$$\rho = \frac{1}{1 + at} \times \text{a function of } p$$
$$= \frac{1}{1 + at} \times \phi(p) \text{ say} \qquad \text{(x)}$$

and equating (ix) and (x)

$$\frac{\phi(p)}{1 + at} = pf(t)$$

i.e. $$\frac{\phi(p)}{p} = (1 + at) f(t)$$

i.e. a function of p alone equals a function of t alone which is only possible if each side equals a constant. Let this constant be $\frac{1}{k}$. Then

$$\phi(p) = \frac{p}{k},$$

and $$f(t) = \frac{1}{k(1 + at)},$$

either (ix) or (x) becomes

$$\rho = \frac{p}{k(1 + at)}$$

i.e. $$p = k\rho(1 + at) \qquad \text{(xi)}$$

giving the relation between p, ρ, t when they are all liable to variation.

It must be realised that Boyle's and Charles' Laws are not absolutely true for all ranges of temperature and pressure. The conception of a "perfect gas" analogous to a "perfect fluid" is introduced for which the gaseous laws are assumed to be true for all ranges. If such a

"perfect gas" were cooled below 0° C., keeping the pressure constant, its volume would theoretically vanish, from (v) at a temperature t given by

$$1 + at = 0,$$

i.e.
$$t = -\frac{1}{a}$$
$$= -273.$$

This temperature of $-273°$ C. is known as the *absolute zero* and temperatures measured from this zero are called *absolute temperatures.* If we let T denote the absolute temperature, then $t°$ C. is equivalent to $273 + t$ on the absolute scale,

i.e.
$$T = 273 + t$$
$$= \frac{1}{a} + t;$$

$\therefore$ Equation (xi) may be written

$$p = k\rho(1 + at)$$
$$= k\rho a\left(\frac{1}{a} + t\right)$$
$$= k\rho a T;$$
$$\therefore \quad \frac{pv}{T} = k\rho a v,$$

where v is the volume of the given mass of gas,

$$= ka . \rho v$$
$$= ka \times \text{mass of the gas}$$
$$= \text{constant, say R,}$$

then
$$pv = RT \quad \text{.......................(xii)}$$

Hence the product of the pressure and the volume of a given mass of gas is proportional to its absolute temperature.

Example 1.—*A kilogram of air is contained in a closed vessel whose capacity is* 1000 *cm.*^{3}. *Find the ratio of its pressure to the atmospheric pressure, assuming the atmospheric density to be* 0·00129 *grm./cm.*^{3}. (*Inter. Sc.*)

The air in the closed vessel has a volume of 1000 c.cm. and a weight of 1 kilog., hence its density is 1 grm./cm.^{3}. But the atmospheric density is 0·00129 grm./cm.^{3};

$\therefore$ ratio of densities is 1 : 0·00129,

or 100,000 : 129;

$\therefore$ from equation (iv),

ratio of pressures is also 100,000 : 129.

Example 2.—*The volume of a certain mass of gas at 10° C. is 250 c cm If the pressure is kept constant, at what temperature will its volume be 300 c cm.?*

From Charles' Law, equation (v),

$$250 = v_0 (1 + 10a),$$

and

$$300 = v_0 (1 + ta),$$

where $t°$ C. is the temperature required

Dividing the second of these equations by the first,

$$\frac{300}{250} = \frac{1 + ta}{1 + 10a},$$

i e

$$6 (1 + 10a) = 5 (1 + ta);$$

$$\therefore \quad t = \frac{1}{5a} + 12$$

$$= 66\ 6° \text{ C. since } a = \tfrac{1}{273}.$$

Example 3.—*A mass of air under a given pressure occupies 24 cub in. at the temperature of 39° C. If the pressure be diminished in the ratio 3 : 4, and the temperature raised to 78° C, show that the volume of the air will be 36 cub. in*

Let p be the original pressure and ρ_1 the original density Then from equation (xi)

$$p = k\rho_1 (1 + 39a) \tag{i}$$

In the second case, let the volume be v, the pressure is given as $\frac{3}{4}p$, and let the density be ρ_2

Then

$$\tfrac{3}{4}p = k\rho_2 (1 + 78a) \tag{ii}$$

Dividing (i) by (ii),

$$\frac{4}{3} = \frac{\rho_1 (1 + 39a)}{\rho_2 (1 + 78a)}$$

$$= \frac{312\, \rho_1}{351\, \rho_2} \text{ since } a = \frac{1}{273}$$

Also, since the mass is constant,

$$24\rho_1 = v\rho_2,$$

$$\therefore \quad \text{we have } \frac{4}{3} = \frac{312}{351} \frac{v}{24},$$

$$\therefore \quad v = 36 \text{ cub in.}$$

76. Mixtures of gases

Suppose two different gases, contained in closed vessels of different volumes, have the same pressure and temperature. If communication is established between the two vessels and no chemical action takes

place between the gases, it is found *experimentally* that the gases permeate each other until they are thoroughly mixed and that then the pressure and temperature are the same as before. This experimental fact enables us to deduce two important results:—

(1) If two gases at the same temperature be mixed together in a vessel of volume v, and if the pressures of the two gases *alone* filling the volume v are p_1 and p_2, then the pressure of the mixture is $p_1 + p_2$.

The first gas has a volume v and pressure p_1.

The second gas has a volume v and pressure p_2,

but if its pressure were p_1, its volume would be $\dfrac{p_2 v}{p_1}$,

since, from Boyle's Law, $\quad p_2 v = p_1 \left(\dfrac{p_2 v}{p_1}\right)$.

Thus we may consider the two gases as having volumes v and $\dfrac{p_2 v}{p_1}$ and both at the pressure p_1.

Hence, from the experimental evidence quoted above, if they are mixed together they form a mixture of total volume $v + \dfrac{p_2 v}{p_1}$ at a pressure of p_1.

Suppose that this mixture has its volume reduced to v, and that its pressure is P.

From Boyle's Law $\qquad p_1 \left(v + \dfrac{p_2 v}{p_1}\right) = \mathrm{P}v;$

$$\therefore \quad \mathrm{P} = p_1 + p_2,$$

which proves the proposition.

(2) Two different gases, volumes v_1 and v_2, pressures p_1 and p_2 respectively, are mixed together so that the volume of the mixture is V. To find the pressure of the mixture if the temperatures are kept constant.

The first gas has a volume v_1 at a pressure p_1.

The second gas has a volume v_2 at a pressure p_2, or, alternatively, a volume $\dfrac{p_2 v_2}{p_1}$ at a pressure p_1.

Hence we may consider that both gases are at a pressure p_1, and have volumes v_1 and $\dfrac{p_2 v_2}{p_1}$.

When they are mixed, the volume of the mixture is therefore

$$v_1 + \frac{p_2 v_2}{p_1} \text{ at a pressure } p_1.$$

But the final volume is to be V, let the necessary pressure be P Then from Boyle's Law

$$PV = p_1 \left(v_1 + \frac{p_2 v_2}{p_1}\right)$$

$$= p_1 v_1 + p_2 v_2,$$

$$\therefore \quad P = (p_1 v_1 + p_2 v_2)/V,$$

and this is the pressure of the mixture

Example 1 —*A vessel containing 2 litres of air at a pressure of $\frac{1}{2}$ atmosphere is put into communication with another vessel containing 3 litres of air at a pressure of 3 atmospheres To find the subsequent pressure of the air in the two vessels*

If the pressure in each mass of air were changed to 1 atmosphere, the air in the first vessel would occupy $2 \times \frac{1}{2}$ litres $= 1$ litre, and that in the second 3×3 litres $= 9$ litres The total mass of air would therefore occupy 10 litres at atmospheric pressure But it has to occupy $2 + 3$ litres $= 5$ litres (the sum of the volumes of the vessels) Since the volume is thus halved, the pressure is doubled therefore the required pressure is 2 atmospheres

Example 2 —*Air of volume 2 cub ft and pressure 3 atmospheres and air of volume 4 cub ft and pressure 2 atmospheres are slowly forced into an originally empty vessel of volume 3 cub ft During the process half the air (by weight) escapes Find the pressure of the air when the process is completed* (*Inter Sc*)

With the notation of the previous paragraph,

$$p_1 = 3 \text{ atmospheres}, \quad v_1 = 2 \text{ cub ft},$$

and $\qquad p_2 = 2 \text{ atmospheres}, \quad v_2 = 4 \text{ cub ft}$

From Boyle's Law, we may consider the second quantity of air to have a pressure 3 atmospheres, when the corresponding volume (V) is given by

$$3V = 2 \cdot 4,$$

$$V = \tfrac{8}{3} \text{ cub ft}$$

Thus we have a combined volume

$$2 + \tfrac{8}{3} \text{ cub ft}$$

all at a pressure p_1 (= 3 atmospheres),

i e $\qquad$ volume of mixture $= \frac{14}{3}$ cub ft,

and $\qquad$ pressure of mixture $= 3$ atmospheres

If the density of the air at this pressure is w lb./cub. ft., then the total weight of air is $\frac{14w}{3}$ lb. and half of this is lost.

Hence final weight is $\frac{7w}{3}$ lb.

i.e. final volume is $\frac{7}{3}$ cub. ft. at a pressure of 3 atmospheres.

Since the volume has to be 3 cub. ft., if P is the corresponding pressure

$$3P = 3 . \tfrac{7}{3} \text{ from Boyle's Law;}$$

$$\therefore \quad P = 2\tfrac{1}{3} \text{ atmospheres.}$$

77. Densities and specific gravities of gases

We set out below a table of specific gravities of the more important gases and the corresponding densities. They are all at a temperature 0° C. and a pressure of 76 cm. of mercury.

SPECIFIC GRAVITIES

Air	0·001293	Nitrogen	0·001256
Oxygen	0·001429	Carbon Dioxide ...	0·001977
Hydrogen	0·0000899	Helium	0·000178

DENSITIES

Air	1·293 grm./litre	=	0·0807	lb./ cub. ft.
Oxygen	1·429 ,,	=	0·0892	,,
Hydrogen	0·0899 ,,	=	0·0056	,,
Nitrogen	1·256 ,,	=	0·0785	,,
Carbon Dioxide	1·977 ,,	=	0·1760	,,
Helium	0·1785 ,,	=	0·0111	,,

78. Decrease of atmospheric pressure with increase of altitude at constant temperature

The pressure intensity of the atmosphere, which is generally about 14·7 lb. wt./sq. in. at the earth's surface, decreases with altitude since at any point (P) above the earth's surface there is less air above P than above the surface and consequently the pressure intensity at P is less than 14·7 lb. wt./sq. in. However, the decrease is not linear, since the air is not homogeneous and consequently its density varies.

By using the integral calculus we may rapidly obtain an expression for the pressure intensity of the atmosphere at a height z above the earth's surface, assuming a constant temperature.

Imagine a column of air [Fig. 133 (*a*)], of cross-section s sq. ft. and suppose PQ represents an element of the column of length δz at a

height of z ft Let the pressure intensity be p at P and $p + \delta p$ at Q and p_0 at S, the earth's surface Let the density of the air be ρ_0 at S and ρ at the height z

Consider the forces acting on the element PQ [Fig 133 (b)] there is a downward force on the upper face of the element of amount $(p + \delta p)\,s$ lb wt

In addition there is the weight of the element of air which is $\rho s \delta z$ lb wt, taking ρ as constant throughout the small range δz

Upwards there is the pressure underneath the element of amount ps lb wt

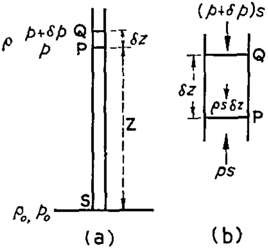


Fig 133

Resolving these forces vertically, for the equilibrium of the element

$$ps = (p + \delta p)\,s + \rho s \delta z,$$

$$0 = \delta p + \rho \delta z,$$

$$\frac{\delta p}{\delta z} = -\rho \qquad (1)$$

Now ρ varies with z, but we know from § 75 that for any gas $p/\rho =$ constant, if the temperature is constant, therefore

$$p/\rho = p_0/\rho_0,$$

and (1) becomes
$$\frac{\delta p}{\delta z} = -\frac{\rho_0}{p_0} p$$

Taking limits as $\delta z \to 0$, this becomes

$$\frac{dp}{dz} = -\frac{\rho_0}{p_0} p,$$

$\therefore$ we may integrate $\displaystyle\int\frac{dp}{p} = -\int\frac{\rho_0}{p_0}dz$

$$= -\frac{\rho_0}{p_0}\int dz \text{ since } \rho_0, p_0 \text{ are constants.}$$

Integrating $\quad \log_e p = -\dfrac{\rho_0}{p_0}z + \text{C}$, where C is the constant of integration.

When $z = o$, $p = p_0$ and therefore

$$\log_e p_0 = \text{C}.$$

Finally $\quad \log_e p = -\dfrac{\rho_0 z}{p_0} + \log_e p_0,$

$$i.e. \quad \log_e\left(\frac{p}{p_0}\right) = -\frac{\rho_0 z}{p_0},$$

$$\text{or} \quad p = p_0 e^{-\frac{\rho_0 z}{p_0}},$$

giving the pressure of the atmosphere (p), at constant temperature at a height z in terms of ρ_0, p_0 the density and pressure of the air at the earth's surface respectively.

In fact, the temperature of the atmosphere does not remain constant at varying heights but decreases with height above the ground at an average rate of 3½° F. per 1000 ft. The rate of fall of temperature is known as the *lapse rate*, the average value of which is very closely the same in all latitudes and at all heights up to a fairly well defined limit which, for England, is about 7 miles. Above this limit the region is known as the *stratosphere*.

If we assume a constant lapse rate we may easily determine the decrease of atmospheric pressure with altitude, taking this change of temperature into consideration. Using the notation of § 75, equation (xi) in that section is

$$p = k\rho(1 + at)$$
$$= k\rho a\text{T}.$$

Hence $\quad \rho = \dfrac{p}{ka\text{T}},$

and equation (i) of this present section becomes

$$\frac{\delta p}{\delta z} = -\rho$$

$$= -\frac{p}{ka\text{T}} \quad \text{(ii)}$$

If, then, the temperature decreases uniformly with the height z,

$$\text{T} = \text{T}_0 - bz,$$

where T_0 is the absolute temperature of the ground and b is a constant (in fact, the decrease of temperature per unit of height)

Thus (ii) becomes
$$\frac{\delta p}{\delta z} = -\frac{p}{ka(T_0 - bz)},$$

and in the limit as $\delta z \to 0$, this is
$$\frac{dp}{dz} = -\frac{p}{ka(T_0 - bz)}$$

Integrating
$$\int\frac{dp}{p} = -\int\frac{dz}{ka(T_0 - bz)}$$
$$= \frac{1}{kab}\int\frac{-b\,dz}{T_0 - bz},$$
$$\log_e p = \frac{1}{kab}\log_e(T_0 - bz) + C \qquad \text{(iii)}$$

C may be determined since $p = p_0$ when $z = 0$,
$$\log_e p_0 = \frac{1}{kab}\log_e T_0 + C,$$

(iii) becomes
$$\log_e\left(\frac{p}{p_0}\right) = \frac{1}{kab}\log_e\left(\frac{T_0 - bz}{T_0}\right)$$
$$= \frac{1}{kab}\log_e\left(\frac{T}{T_0}\right)$$

which is a relation between the atmospheric pressure at any height, the temperature there, the pressure and temperature at ground level and known constants

Example —*To find the atmospheric pressure at a height of 10 000 ft, assuming the temperature remains constant throughout this height*

The atmospheric pressure at the earth's surface (p_0) is 14 7 lb wt /sq in, i e 14 7 × 144 lb wt /sq ft, and from the table in § 77, the density of air at the earth's surface (ρ_0) is 0·0807 lb / cub ft

Thus
$$p = p_0 e^{-\frac{\rho_0 z}{p_0}}$$
$$= 14\ 7e^{-\frac{0{\cdot}0807 \times 10\,000}{14\ 7 \times 144}}\ \text{lb wt /sq in}$$
$$= 14\ 7e^{-0{\cdot}38}\ \text{lb wt./sq in}$$
$$= 10{\cdot}05\ \text{lb wt /sq in from tables}$$

Exercises X

(1) If the atmospheric pressure is 15 lb. per square inch and the diameters of a pair of Magdeburg Hemispheres are 7 in., find the force required to pull them apart.

(2) If the atmospheric pressure at the surface of the earth be $14\frac{1}{2}$ lb. per square inch, find the height of the water barometer in feet.

(3) The section of the closed limb of a siphon barometer is to that of the open limb as 3 to 17. The mercury rises 1·275 in. in the closed branch. What change takes place in the mercury of an ordinary barometer?

(4) A siphon barometer is so constructed that the long closed tube has an internal sectional area equal to $\frac{1}{4}$ sq. in., while the short open tube has an internal sectional area equal to $\frac{1}{2}$ sq. in. Find what fall will take place in the long tube of this barometer when the true pressure of the air falls 1 in.

(5) A body floats in water contained in a vessel placed under an exhausted receiver with half its volume immersed. Air is then forced into the receiver till its density is 80 times that of air at atmospheric pressure. Show that the volume immersed in water will then be four-ninths of the whole volume, assuming the specific gravity of air at atmospheric pressure to be 0·00125.

(6) In a tube of uniform bore a quantity of air is enclosed. What will be the length of this column of air under a pressure of 3 atmospheres, and what under a pressure of $\frac{1}{3}$ atmosphere, its length under the pressure of a single atmosphere being 12 in.?

(7) Mercury is poured into a uniform bent tube, open at both ends, and having its two branches vertical. One end is closed, its height above the mercury being 4 in. How much mercury must be poured into the open end so that the mercury may rise 1 in. in the closed branch?

(8) A straight uniform tube closed at one end, whose length is $2h$, has the open end just immersed in a basin of mercury. If the tube contains a quantity of air which under atmospheric pressure would occupy a length of the tube equal to $\frac{4}{9}h$, show that the mercury will rise in the tube to a height equal to $\frac{2}{3}h$, h being the height of the mercury barometer at the time of the experiment.

(9) A bubble of air whose volume is 0·0004 cub. in. is formed at the bottom of a pond 17 ft. deep. What will be its volume when it reaches the surface?

(10) When the reading of the true barometer is 30 in., the reading of a barometer the tube of which contains a small quantity of air, and whose height above the surface of the mercury in which it is immersed is $31\frac{1}{3}$ in., is 28 in. If the reading of the true barometer falls to 29 in., show that the reading of the faulty barometer will be $27\frac{1}{3}$ in.

(11) If the atmosphere were homogeneous of density 0 0807 lb /cub ft, find what its height would be if the atmospheric pressure at the earth's surface is 14 7 lb wt /sq in

(12) 100 cub in of air at a pressure of 15 lb to the square inch are pumped into a chamber already containing 50 cub in of air at a pressure of 10 lb to the square inch What is the pressure of the mixture?

(13) The top of a uniform cylindrical barometer tube is 33 in above the level of the mercury in the container There is a small quantity of air imprisoned above the mercury at the top of the tube so that the barometer reads 28 6 in. when it should read 29 in What should the barometer read when the actual reading is 29 48 in? (Inter Sc)

(14) A mass of air under given pressure occupies 44 cub in at a temperature of 13° C If the volume of the air is reduced to 24 cub in, and the temperature raised to 39° C, show that the pressure will be doubled

(15) A gas whose volume is 3 and whose pressure is that of 27 in of mercury is mixed with a gas whose volume is 2 5 and pressure that of 32 in of mercury If the original temperatures of the gases were the same what volume will the mixture occupy under a pressure of 30 in of mercury, the temperature being unaltered?

(16) When the height of the barometer is 30 in, a glass tube 20 in long, sealed at the upper end, is immersed vertically in mercury until the upper end of the tube is in the level of the surface of the mercury Find how far the mercury will rise in the tube (Inter Sc)

(17) Determine the buoyancy of the air on a spherical balloon 10 metres in diameter, given that a metre cube of air weighs one kilogram, and, if the balloon is filled with hydrogen of which 8 cubic metres weigh a kilogram, prove that it can rise if the gross weight is about 458 kilog

(18) Air at a pressure of 20 lb /in² is imprisoned in a cylinder which is fitted with a piston When the piston is pushed in a distance of 8 in from its initial position the pressure of the contained air becomes 30 lb /in² Calculate the pressure when the piston is pushed in 12 in from its initial position, and the thrust then exerted on the piston if its section is a circle of diameter 6 in (Inter Sc)

(19) A cylindrical jar, of length $3a$, open at the top, is inverted and partially immersed so that one third of its length is under water If the height of the water barometer is twice the length of the jar, find the distance which the water rises in the jar

(20) A heavy cubical vessel, made of thin sheet metal, is of side 3 ft and has a small hole in one face It floats half immersed on the surface of a large sheet of water with this face horizontal and lowermost, the vessel itself being one-third full of water If it is slowly drawn down under the surface without rotation show that, after its top face has reached a certain depth below the surface, it will not rise again if released Taking the height of the water barometer to be 34 ft, find this depth (H S C, I)

ANSWERS

1. $\frac{1}{2}$ atmosphere. 2. 33·408 ft. 3. $1\frac{1}{2}$ in. rise.
4. $\frac{2}{3}$ in. 6. 4 in., 36 in. 7. Enough to fill 12 in. of tube.
9. 0·0006 cub. in. 11. 4·97 miles. 12. 40 lb. wt./sq. in.
13. 29·98 in. 15. 5·36. 16. 6·28 in.
17. $\frac{500\pi}{3}(1 - s)$ where s is S.G. of hydrogen compared with air.
18. 40 lb. wt./sq. in., 1131·12 lb. wt. 19. $0{\cdot}31a$. 20. $3\frac{19}{28}$ ft.

CHAPTER XI

HYDROSTATIC MACHINES

79 Introduction

We propose to consider in this chapter the action of machines which depend for their working on hydrostatic principles One such machine, the Hydraulic (Bramah) Press, we considered in § 10 and here we shall study the action of the diving bell, siphon and a variety of pumps

80 The diving bell

This is a heavy cylindrical or bell shaped vessel, closed at the top and open at the bottom It is large enough to hold several people and heavy enough to sink in water under its own weight, taking its contained air down with it It is lowered by means of a chain and its purpose is to enable men to work in it at the bottom of deep water As it sinks into the water, the pressure of the contained air gradually increases since it is equal to the pressure of the water with which it is in contact Hence the air is compressed and water rises slightly in the bell To overcome this, there are two tubes connecting the bell with the surface, through one of which air is pumped while through the other, air can escape The bell is supported partly by the tension of the chain and partly by the buoyancy of the water, which depends on the volume of air in the bell

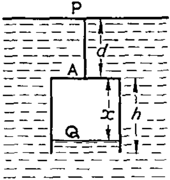


Fig 134

Suppose the bell represented diagrammatically in Fig 134, is cylindrical and no air is pumped in or allowed to escape To determine for a certain depth —

(1) The height to which water rises in the bell,

(2) The tension in the supporting chain,

(3) The volume of air at atmospheric pressure which would have to be forced in to prevent the water rising in the bell

(1) Let H be the height of the water barometer, s sq ft the cross section area of the bell, d ft, the depth of the top of the bell, h ft, the height of the bell and x ft, the height of free air in the bell

Originally the volume of air in the bell $= hs$ cub. ft., and the original pressure is atmospheric (H ft. of water).

The pressure of the air in the bell when it is submerged is equal to the pressure intensity of the water with which it is in contact,

i.e. $\quad = $ pressure intensity at depth $(d + x)$ ft.

$$= \mathrm{H} + d + x,$$

and the volume, when submerged,

$$= sx \text{ cub. ft.}$$

Therefore, from Boyle's Law,

original pressure $\times$ volume $=$ final pressure $\times$ volume,

i.e.
$$\mathrm{H}.hs = (\mathrm{H} + d + x)\, sx;$$

$$\therefore \quad x^2 + x(\mathrm{H} + d) - \mathrm{H}h = 0.$$

This is a quadratic equation for x with only one positive root, which is the value we require, when $h - x$ is the height to which water rises in the bell.

(2) To find the tension in the supporting chain, we equate the downward forces acting on the bell to the upward forces. The downward forces are:—

(i) Weight of bell. Let this be W lb. wt.

(ii) Weight of enclosed air.

The upward forces are:—

(i) Tension in the chain, say T lb. wt.

(ii) The upthrust due to the water displaced by the bell.

The amount of water displaced by the bell is xs cub. ft., therefore its weight is $xs\rho$ lb. wt. if ρ is the density of water. We may neglect the upthrust due to the water displaced by the actual metal of the bell, since this will be very small in comparison with the air displaced, and we may also neglect the weight of the enclosed air which, again, will be very small compared with the weight of the bell.

Hence, equating upward and downward forces,

$$\mathrm{T} + xs\rho = \mathrm{W};$$

$$\therefore \quad \mathrm{T} = \mathrm{W} - xs\rho \text{ lb. wt.}$$

(3) Let the volume of air which must be forced in at atmospheric pressure (H) to expel the water from the bell be v. The volume of air originally in the bell at atmospheric pressure was hs.

Thus we have a total volume $v + hs$ of air at atmospheric pressure H compressed into a volume hs at a pressure of $\mathrm{H} + h + d$;

$\therefore$ from Boyle's Law

$$(v + hs)\,\mathrm{H} = hs\,(\mathrm{H} + h + d);$$

$$v = \frac{hs\,(h + d)}{\mathrm{H}}.$$

Example 1—*An iron diving bell weighs 6 ton, and holds 160 cub ft of air. To find the tension on the supporting chain when the bell is completely immersed in sea water and kept full of air (specific gravity of iron = 7 2, of sea water 1 024)*

Weight of a cubic foot of sea water = 1024 oz = 64 lb wt,

weight of water displaced by air inside = 64 × 160 = 10 240 lb,

" " " iron of bell = 6 × 1 024 — 7 2 ton

= 1911 lb (to nearest lb),

total weight of water displaced = 12 151 lb wt

But weight of bell = 6 × 2240 = 13,440 lb wt,

tension of chain = 1289 lb wt

Example 2—*If a bell whose internal capacity is 200 cub ft is lowered in a river till its base is 20 ft below the surface, to find how many cubic feet of air at atmospheric pressure must be pumped in to prevent the water from rising inside*

Let v be the volume which the air filling the bell would occupy when at atmospheric pressure

The pressure of atmosphere = that due to 34 ft of water,

inside bell = 34 + 20 ft

The volume actually occupied by the air = 200 cub ft

Therefore by Boyle's Law

$$v \times 34 = 200 \times 54,$$

$$v = 200 \times 54 - 34 = 318 \text{ cub ft nearly}$$

Hence 318 − 200 or 118 cub ft of air at atmospheric pressure must be pumped in

Example 3—*A bottle full of air is inverted and lowered in water to a depth of 51 ft To find how much water has entered the bottle*

Here the pressure increases 1 atmosphere for 34 ft, or $1\frac{1}{2}$ atmospheres for 51 ft descended Therefore the pressure at 51 ft depth is $2\frac{1}{2}$ or $\frac{5}{2}$ times that at the surface Hence by Boyle's Law, the volume of the air is $\frac{2}{5}$ its volume at the surface Therefore the water enters till it fills the remaining $\frac{3}{5}$ of the volume of the bottle

Example 4—*A cylindrical diving bell 9 ft high is lowered into a lake until the top of the bell is 11 ft below the surface If no air is pumped in to find how high the water rises in the interior*

Let x ft be the height still occupied by air (AQ Fig 134)

Then the depth PQ = $(11 + x)$ ft

The pressure at Q is therefore that due to a head of water of

$$(34 + 11 + x) \text{ ft.} = (45 + x) \text{ ft.}$$

But the air originally occupied a length of 9 ft. under a pressure of 34 ft. head of water. Therefore, by Boyle's Law,

$$34 \times 9 = (45 + x) \times x;$$
$$\therefore \quad x^2 + 45x - 306 = 0.$$

Solving this quadratic equation by factorising or otherwise, we have

$$(x + 51)(x - 6) = 0;$$
$$\therefore \quad x = -51 \text{ or } 6.$$

Now the length occupied by air cannot be a *minus* quantity; $\therefore \quad x = 6$, and the water rises in the bell through $9 - 6$ or 3 ft.

Example 5.—*A cylindrical diving bell of uniform cross-section is suspended vertically with its base just immersed in a large lake and the air supply line is then closed by a valve at its junction with the bell. The diving bell has a height h equal to that of the water barometer. If the bell is lowered a distance nh, find the fraction λ of the length of the bell which contains air and show that, when n is very large, $\lambda = 1/n$ approximately.*

If additional air is then pumped in to clear the bell of water, show that this is sufficient to fill n such diving bells in the original position with air at atmospheric pressure (n is not necessarily taken large here). (*H.S.C., I.*)

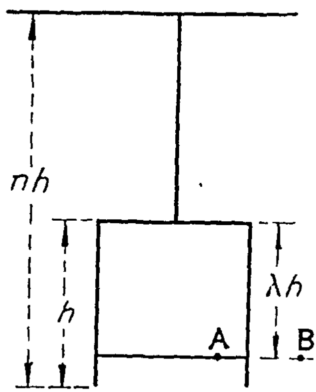


Fig. 135.

Consider Fig. 135. If A and B represent two points at the same level, A being in the surface of the water in the diving bell, the pressure intensities at A and B are equal. But the pressure intensity at A is the pressure to which the air in the bell is subjected, and

press. int. at B = atmospheric pressure + press. due to depth of B

$$= h + (n - 1)h + \lambda h,$$

since h is the height of the water barometer and the depth of B is $nh - h + \lambda h$;

$$\therefore \quad \text{pressure on air in bell} = h + (n - 1)h + \lambda h.$$

But volume of air in bell $= \lambda hs$ if s = cross-section area.

Originally the volume was hs and the pressure h, thus from Boyle's Law

$$h . hs = \lambda hs[h + (n - 1)h + \lambda h];$$
$$\therefore \quad 1 = \lambda(1 + n - 1 + \lambda)$$
$$= \lambda n + \lambda^2;$$

∴ λ is given by

$$\lambda^2 + \lambda n - 1 = 0 \qquad (i)$$

$$\therefore \lambda = -\tfrac{1}{2}n + \tfrac{1}{2}\sqrt{n^2+4},$$

since only the positive value of the square root can be taken to give a positive value for λ

It is clear from equation (i) that when n is large, λ is a small positive fraction and hence λ^2 is negligible compared with λ, so that for very large values of n, equation (i) becomes approximately

$$\lambda n = 1,$$

i e $$\lambda = \frac{1}{n}$$

Suppose that a volume V of air at atmospheric pressure h has to be pumped in to clear the bell of water Originally the bell contained a volume hs of air at atmospheric pressure h, thus there is a total volume,

$V + hs$ of air at atmospheric pressure h,

but this has been condensed to a volume

hs of air at a pressure equal to the pressure intensity at a depth nh,

i e a pressure intensity of $h + nh$

Thus, from Boyle's Law

$$(V + hs)\,h = hs\,(h + nh)$$
$$V = snh$$
$$= n\,sh,$$

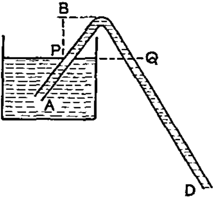


Fig 136

i e the volume of air at atmospheric pressure required to be pumped in is n times the volume of air contained by the bell in the original position at atmospheric pressure,

i e the volume required is sufficient to fill n such diving bells in the original position with air at atmospheric pressure

81. The siphon

This is an instrument which may be used for emptying vessels containing liquid It consists of a bent tube, one arm being longer than the other (Fig 136)

To explain its action, suppose that the siphon has been filled with liquid, both ends A, D having been temporarily closed with plugs, and that the shorter arm has been lowered into a vessel of the same liquid, as in Fig 136 Now let the end A be opened, the end D being still closed

Then, if the height PB is less than the height to which the liquid would ascend in a barometer, the pressure of the atmosphere on the surface P will prevent a vacuum from forming in the tube, which will therefore remain filled with liquid.

And if Q be taken on the longer arm on the level of the surface at P, then (by connecting Q with P by a zigzag or horizontal and vertical lines) we may show that the pressure at Q is equal to that at P, *i.e.* to the atmospheric pressure. The pressure inside the tube at D is therefore greater than outside by the amount due to the column QD, and this excess of pressure tends to force the plug out.

If, therefore, the plug is removed, the liquid will flow out at D. And, since no vacuum is formed in the tube, the pressure of the atmosphere at P will cause fresh liquid to rise in the tube at A, thus producing a continuous stream.

Thus the two conditions which are necessary for the siphon to function are:—

(1) The end D must be below the level of the liquid in the vessel which is to be emptied, *i.e.* D must be lower than P, and

(2) The height of the top of the siphon above the level of the liquid must be less than the height of the barometer for the liquid concerned, *i.e.* BP must be less than about 34 ft. if water is being siphoned or less than about 30 in. if mercury is siphoned.

82. Pumps

The remainder of this chapter is devoted to pumps and we shall consider those which depend for their working on the laws of fluid pressure.

The types of pumps here studied will be -

(i) Water Pumps, for raising water and discharging it at a higher level;

(ii) Air Pumps, for exhausting or partially exhausting the air from a vessel;

(iii) Air Condensers, for increasing the pressure of air in a vessel.

83. The common or suction pump

This consists of a barrel or cylinder connected with the water to be raised by a pipe which opens into the bottom of the barrel (Fig. 137). There is a valve U, opening upwards, where the pipe joins the barrel.

In the barrel there is a closely fitting piston (P) which can be raised or lowered by means of a piston rod connected to the pump handle (L). There is another valve V, opening upwards, in the piston and the water is discharged through the spout S.

To explain the action of the pump start with the barrel full of water and the piston at the bottom of the cylinder

In the up stroke [Fig 137 (*a*)] the valve V remains closed and the pressure below the piston is reduced and the atmospheric pressure acting on the surface of the water in the well forces water up the pipe which lifts the valve U and enters the barrel At the same time the water above the piston is raised to the level of the spout and runs out

In the down stroke [Fig 137 (*b*)] the valve U closes and the water lifts the valve V and passes from the lower to the upper side of the piston P

In the next up stroke this water is raised to the spout while a fresh supply of water runs into the barrel through the valve U

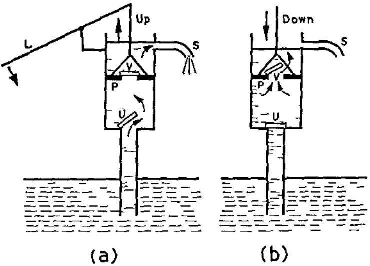


Fig 137

Since the water below the piston is raised from below by the pressure of the atmosphere it follows that the height of the piston above the surface of the water must never exceed the height of the water barometer otherwise a vacuum will be formed in the barrel and water will cease to flow in

When the pump is first placed in water the pipe and barrel are full of air which has to be pumped out before the water will rise in the barrel

Suppose we start with the piston at the bottom of the barrel

In the first up-stroke the air in the pipe expands and part of it rushes through the valve U into the barrel while the reduction of pressure allows a column of water to rise up into the pipe

In the first down-stroke, the valve U closes, and as soon as the air in the barrel is compressed to atmospheric pressure it begins to escape through V.

In the next up-stroke, the air in the pipe again expands through the valve U into the cylinder, and the reduction of pressure allows the water to rise still further in the pipe. This process continues till the water at last reaches the barrel, when the continuous action as a water-pump begins, and a volume of water equal to that of the barrel is raised at each stroke.

Example 1.—*To find the force required to lift the piston (neglecting the weight of the piston), if its sectional area is 100 sq. cm. and the spout is 10 m. above the water surface in the well.*

Let x cm. be the depth of the piston below the spout, h cm. the height of the water barometer. Then the pressures above and below the piston are due to heads of water of heights

$$(h+x) \text{ and } \{h-(1000-x)\} \text{ cm.},$$

respectively. Therefore their difference is that due to a head of 1000 cm. (the total height of the column, as we should expect). Hence

$$\text{diff. of pressures on two sides of the piston} = 1000 \text{ grm. per sq. cm.}$$

Also $$\text{area of piston} = 100 \text{ sq. cm.};$$

$$\therefore \quad \text{resultant force on piston} = 1000 \times 100 \text{ grm.} = 100 \text{ kilog.}$$

Example 2.—*If the spout is 10 ft. above the water surface, and 5 lb. of water are delivered at each stroke, to find the work done in the up-stroke.*

Let the length of the stroke be l ft., and let the sectional area of the piston be A sq. ft.

The difference of pressures on the two sides of the piston

$$= \text{that due to a head of 10 ft. of water}$$
$$= 10{,}000 \text{ oz. per sq. ft.}$$
$$= 10{,}000/16 \text{ lb. per sq. ft.};$$

$$\therefore \quad \text{resultant thrust on piston} = 10{,}000 \times \text{A}/16 \text{ lb.};$$

$$\therefore \quad \text{work done in up-stroke} = 10{,}000 \times \text{A}l/16 \text{ ft.-lb.}$$

Now $$\text{A}l = \text{volume of water raised to spout in cubic feet};$$

$$\therefore \quad 1000\text{A}l = \text{weight of water raised in ounces,}$$

and $$1000\text{A}l/16 = \text{weight of water raised in pounds}$$
$$= 5 \text{ lb. (by data)};$$

$$\therefore \quad \text{work done in up-stroke} = 5 \times 10 \text{ ft.-lb.} = 50 \text{ ft.-lb.}$$

This is the work required to raise the 5 lb. of water through the total height of 10 ft.

Hence the work done by the pump is the same as if the water were lifted directly up from the bottom of the well to the spout This is in accordance with the Principle of Conservation of Energy

Example 3 —*The bottom of the barrel is 20 ft above the surface of the water and the section of the pipe is $\frac{1}{5}$ of that of the barrel find the height of the water when the piston has been raised 1 ft given the height of the water barometer = 33 ft*

Let x ft be the required height of the water

Before the up stroke the air occupies 20 ft of pipe under a pressure of 33 ft of water

After the up stroke the air occupies $(20 - x)$ ft of pipe *plus* the volume of the barrel under a pressure of $(33 - x)$ ft of water

Now the volume of air in the barrel is 5 times the volume of an equal length of pipe and is therefore equal to that of 5 ft of pipe

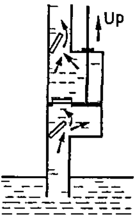


Fig 138

Hence the air occupies a total volume equivalent to

$$(20 - x + 5) \text{ ft}$$

i e $\quad 25 - x$ ft of pipe

Therefore from Boyle s Law

$$20 \times 33 = (25 - x)(33 - x)$$

$$x^2 - 58x + 165 = 0$$

$$x = 55 \text{ or } 3$$

The only possible solution is $x = 3$ and the water rises 3 ft

84 The lifting pump

This is an adaptation of the common pump for raising the water above the pump to any desired height

The top of the barrel (Fig 138) is not open as in the common pump but has a lid through which the piston rod passes by means of a tight fitting collar From the barrel a pipe leads upwards with a valve at the junction

In the up stroke the water above the piston is lifted up into the upper pipe and more water is drawn up into the barrel below the piston

In the down stroke the upper and lower valves close the middle one (in the piston) opens and allows the water below the piston to pass through into the upper part of the barrel

There is no limit to the height to which water can be lifted above the piston but as in the common pump the column below the piston cannot exceed the height of a barometer of the liquid that is being pumped

85. The forcing pump

This differs from the common pump in having a solid piston [Fig. 139 (*a*)], and the outlet pipe leads away from the bottom of the barrel, there being a valve (V) at the join.

In the up-stroke, the valve V closes, the valve U opens and water is drawn into the barrel.

In the down-stroke [shown in Fig. 139 (*a*)] the valve U closes, V opens and water is forced through the outlet pipe. This results in an intermittent flow, since no water is forced through the outlet pipe when the piston is being raised. This is, for many purposes, a serious disadvantage and this may be overcome to a certain extent by making the water pass through the valve V at the bottom of the barrel into a chamber (C) [Fig. 139 (*b*)], from which the outlet pipe emerges. Thus, when the water in this chamber reaches the base of the outlet tube a certain amount of air is imprisoned. If water is pumped up faster than it can get away through the outlet pipe the air in the chamber is compressed and this pressure helps to maintain an even flow through the outlet tube while the piston is being raised.

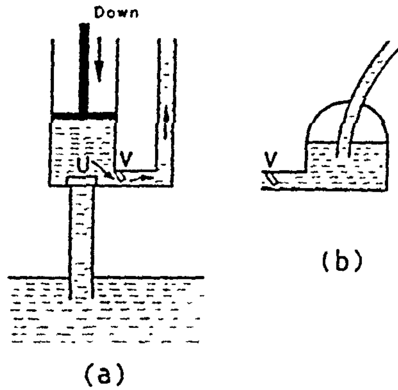


Fig. 139.

86. The air pump

This is simply a common pump used for pumping out air instead of water. The vessel to be exhausted of air is called the receiver (A) and the pump itself consists of a cylinder (B) in which a piston travels, both containing valves opening outwards from the receiver.

To describe the action of the pump, consider Fig. 140, and suppose the piston at the bottom of the barrel.

In the up-stroke [Fig. 140 (*a*)] the valve V closes, and the air in the receiver and tube lifts the valve U, and part of it passes into the barrel. At the end of the up-stroke the barrel is therefore filled with air at the same pressure, and therefore also at the same density, as the air left in the receiver.

In the first part of the down-stroke [Fig. 140 (*b*)] the valve U closes, *and the valve V also remains closed, while the air beneath the piston is compressed until its pressure equals that of the atmosphere.*

In the remainder of the down stroke [Fig 140 (c)] the piston valve V opens and allows the air to escape from beneath the piston

To find the density and pressure of the air left in the receiver after n strokes

Let A be the volume of the receiver and connecting pipe B that of the barrel Let D be the density of the atmospheric air originally in the receiver Let d_1 d_2 d_n be the densities of the air left after 1 2 n strokes respectively

After the first up stroke the air originally in the receiver expands from volume A to volume A + B Hence since its mass is unaltered its densities are connected by the relation

$$d_1 (A + B) = DA$$

$$d_1 = D \frac{A}{A + B}$$

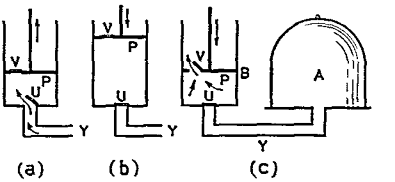


Fig 140

During the down stroke the air left in the receiver remains at the same density d_1 unaltered but in the next up-stroke it again expands in volume from A to A + B Hence for its subsequent density we have

$$d_2 (A + B) = d_1 A$$

$$d_2 = d_1 \frac{A}{A + B} = D \left(\frac{A}{A + B}\right)^2$$

At the third stroke the air left in the receiver again expands in volume from A to A + B and therefore

$$d_3 (A + B) = d_2 A$$

$$d_3 = d_2 \frac{A}{A + B} = D \left(\frac{A}{A + B}\right)^3$$

Proceeding in this way, it is obvious that the density of the air is reduced at each up-stroke in the ratio of A to A + B, and therefore after n strokes it is given by

$$d_n = D\left(\frac{A}{A+B}\right)^n.$$

But since $$\frac{\text{pressure in receiver after } n \text{ strokes}}{\text{atmospheric pressure}} = \frac{d_n}{D},$$

we have

$$\text{pressure in receiver after } n \text{ strokes} = \left(\frac{A}{A+B}\right)^n \text{atmospheres.}$$

Example 1.—*The volumes of the barrel and receiver are* 25 *and* 75 *cub. in.; to find the pressure of the air left after* 3 *strokes.*

In the first up-stroke, 75 cub. in. of air at atmospheric pressure expand till they fill the receiver and barrel, *i.e.* 100 cub. in.;

$$\therefore \quad \text{pressure after the stroke} = \tfrac{75}{100} = \tfrac{3}{4} \text{ atmosphere.}$$

In each succeeding up-stroke, the air remaining in the receiver expands from 75 to 100 cub. in., and its pressure is therefore reduced to $\frac{3}{4}$ what it was before;

$$\therefore \quad \text{pressure after 3 strokes} = \tfrac{3}{4} \times \tfrac{3}{4} \times \tfrac{3}{4} = (\tfrac{3}{4})^3 = \tfrac{27}{64} \text{ atmosphere.}$$

Example 2.—*The volume of the barrel being two-fifths that of the receiver, to find how many strokes are required to reduce the density to less than one-third the original density.*

$$\text{Here} \qquad B = \frac{2}{5}A; \qquad \therefore \quad \frac{A}{A+B} = \frac{5}{5+2} = \frac{5}{7}.$$

Now $(\frac{5}{7})^2 = \frac{25}{49} > \frac{1}{3}$, $(\frac{5}{7})^3 = \frac{125}{343} > \frac{1}{3}$, $(\frac{5}{7})^4 = \frac{625}{2401} < \frac{1}{3}$.

Hence 4 strokes are required.

87. The condenser or condensing pump

This is, in effect, the ordinary bicycle pump; used to increase the pressure of the air in a vessel or receiver. It consists of a barrel B, traversed by a piston P, and communicating at one end with the vessel A, into which air is to be compressed.

This vessel is called the *receiver*, and is shown in Fig. 141.

Both the piston and the end of the barrel contain valves V, F opening *from* the outside air *towards* the receiver.

In the backward stroke (*i.e.* when the piston P is being pulled back [Fig. 141 (*a*)], the valve F is closed by the pressure in the receiver, while air at atmospheric pressure passes through the valve V to the front of the piston and fills the barrel.

In the beginning of the forward stroke [Fig 141 (*b*)] both valves V, F remain closed, and the air inside the barrel is compressed until its pressure just equals that in the receiver, except in the first stroke, when the air in the receiver is at atmospheric pressure and F opens at once

In the remainder of the forward stroke [Fig 141 (*c*)] the valve F opens, and air is forced through it into the receiver

In what follows, the backward and forward strokes of the piston of a pump are together considered as constituting one complete stroke of the pump

Example 1 —*The volume of the receiver is 80 cub in, and that of the barrel 20 cub in Find how many strokes must be made before the pressure of the air in the receiver is 3 atmospheres*

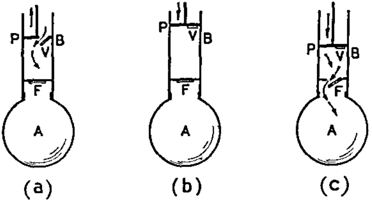


Fig 141

By Boyle's Law the density in the receiver is three times the density of atmospheric air Hence the air in the receiver would occupy 240 cub in at atmospheric pressure,

160 cub in of air have been forced in

But at each back stroke 20 cub in of air enter the barrel and are forced into the receiver at the forward stroke,

number of complete strokes $= 160/20 = 8$

Example 2 —*To find when the valve in the barrel opens in the next forward stroke* (*see Example* 1)

The valve F opens when the air in the barrel has a pressure of 3 atmospheres, that is, when it occupies one-third its original volume or the piston has traversed two-thirds the length of the barrel

To find the density and pressure in the receiver after n complete strokes.

Let A be the volume of the receiver, B that of the barrel, D the density of atmospheric air, d the density in the receiver after n strokes.

Then the receiver originally contained a mass of air AD.

At each backward stroke a volume B of air at atmospheric density D enters the barrel. At the forward stroke this air enters the receiver. Hence, after n *complete* strokes,

$$\text{mass of air in receiver} = (\mathrm{A} + n\mathrm{B})\ \mathrm{D}.$$

But its volume $= \mathrm{A}$;

$$\therefore \quad \text{its density } d = \frac{\mathrm{A} + n\mathrm{B}}{\mathrm{A}}\ \mathrm{D} = \left(1 + n\frac{\mathrm{B}}{\mathrm{A}}\right)\mathrm{D}.$$

Hence

$$\text{pressure in receiver} = \left(1 + n\frac{\mathrm{B}}{\mathrm{A}}\right)\text{atmospheres}.$$

Exercises XI

(1) A siphon is filled with water and inverted into a vessel of liquid of specific gravity 1·6. What is the condition that the liquid may flow through the siphon?

(2) A diving bell whose capacity is 500 cub. ft. is lowered in water until its mouth is at a depth of 51 ft. below the surface. How much air at ordinary atmospheric pressure must be pumped in so that all the water may be expelled?

(3) The top of a cylindrical diving bell, whose volume is 200 cub. ft. and height 8 ft., is at a depth of 60 ft. below the surface of the water. How much air at ordinary atmospheric pressure must be pumped in to keep the bell full of air?

(4) A diving bell 8 ft. high is lowered in water until its top is 60 ft. below the surface. What depth of water will have entered the bell?

(5) A diving bell is lowered in a lake until two-thirds of it is filled with water. Show that, if d be the depth of the top of the bell below the surface, the height of the bell is $3\ (2h - d)$, where h is the height of the water barometer.

(6) If the water barometer stands at 33 ft. 8 in., and if a common pump is to be used to raise petroleum from an oil-well, find the greatest height at which the lower valve of the pump can be placed above the surface of the oil in the well. (The specific gravity of petroleum is 0·8.)

(7) The volumes of the receiver and barrel of a condenser are in the ratio of 5 to 1; find the density of the air in the receiver after 3 complete strokes.

(8) The volume of the receiver of an exhausting air pump being 9 times that of the barrel, how many strokes must be made before the density in the receiver is one-third that of the external air?

(9) A diving bell of cylindrical form is sunk to a given depth, how much must the temperature of the enclosed air be raised that the water may not rise inside the bell?

(10) A cylindrical tumbler, the volume and density of whose substance are v and w', and the area of whose internal cross section is A, is forced mouth downwards into a liquid of density w prove that it will tend to sink when the length of the air column inside it reaches the value

$$\frac{v}{A}\left(\frac{w'}{w}-1\right)$$

(11) Calculate the height to which the water will rise in a cylindrical diving bell, 12 ft high, when its top is lowered at a depth of 60 ft, being given that the temperature of air at the surface is 27° C, height of water barometer at surface 33 ft, and temperature of water 7° C

(12) A barometer tube 1 ft long, containing air at atmospheric pressure, was lowered vertically into the sea with its closed end uppermost When it reached the surface again it was found that the water had risen to a maximum height of 10 in in the tube Taking the height of the mercury barometer as 30 in, the specific gravity of mercury as 13 6 and the specific gravity of sea water as 1 025, find the depth reached by the tube (H S C, I)

(13) A uniform metal pipe of small bore is closed at one end and bent into the form of three quarters of the circumference of a circle It is placed in the vertical plane so that the closed end is at the highest point of the circle, and liquid is poured in until it is on the point of overflowing If the imprisoned air then occupies half the length of the pipe, which was originally 16 ft in length, find the specific gravity of the liquid Assume the water barometer to stand at 33 ft

(14) A motor car has its centre of gravity equidistant from the two axles If each tyre has an area of 25 sq in in contact with the ground find the pressure of the air in each tyre

If a tyre has a volume, when inflated of 2000 cub in and originally contains no air, find how many complete strokes of a pump must be given to inflate the tyre to the required pressure, the internal volume of the pump being 25 cub in Take atmospheric pressure as 14 5 lb wt /sq in and neglect the volume of the connection

(15) A diving bell is in the form of a circular cylinder of diameter h and height h, surmounted by a hemisphere of equal diameter If it is lowered into water and no air is pumped in, find the depth of the top of the bell when the water just fills the cylinder Show that to expel the water from the cylinder the volume of air at atmospheric pressure that must be pumped in is $(3 + h/H)$ V, where V is the volume of the bell and H is the height of the water barometer

(16) The volume of the receiver of an air pump is A and that of the barrel is B A body when placed under the receiver weighs w oz, and after n strokes weighs w_1 oz Determine the weight of the body in a vacuum

Also, if the density of the body is ρ, find the density of the air in the receiver at first

(17) A receiver of capacity R has two barrels connected with it; one condensing, the other exhausting. The condensing barrel has a capacity a and the exhausting barrel a capacity b. They take their strokes alternately, beginning with the condenser; find the density of air in the receiver after n strokes of both, in terms of the atmospheric density D.

(18) A diving bell is in the form of a hollow paraboloid, 10 ft. long, and it is sunk in water with its open end downwards until the water rises 5 ft. internally. Find the depth of the vertex below the surface of the water, taking the water barometer as 33 ft.

(19) An inverted hemisphere, of radius r, is full of air and is forced down so that it is just immersed in mercury. If the height of the mercury barometer is h, and if the mercury rises a distance x in the hemisphere, show that x is given by the equation

$$x^4 - (h + r)x^3 - 3r^2x^2 + (3r^2h + 5r^3)x - 2r^4 = 0.$$

(20) A heavy air-tight piston can slide freely in a vertical cylinder containing air. Initially the piston is at rest and the air below it is at atmospheric pressure, the length of the cylinder containing air then being h. The piston is allowed to descend. Show that it comes to instantaneous rest again at a distance x from the bottom, given by the equation

$$\log_e\left(\frac{x}{h}\right) = \frac{h - x}{d},$$

where d is the distance from the bottom at which the piston would rest in equilibrium. Assume that Boyle's Law holds and that there is no change in temperature.

ANSWERS

1. The highest point must be less than $21\frac{1}{4}$ ft. above the level of the liquid in the vessel.

2. 750 cub. ft. 3. 400 cub. ft. 4. 5·19 ft.

6. 42 ft. 1 in. 7. 1·6. 8. 11.

9. $\frac{h}{H}(273 + t)°$, where H is height of water barometer, h depth of foot of bell, and $t°$ the original temperature.

11. 8·2 ft. 12. 165·7 ft. 13. 13·746.

14. 28 lb. wt./sq. in., 155. 15. $3H - \frac{1}{2}h$.

16. $\dfrac{w_1(A + B)^n - wA^n}{(A + B)^n - A^n}$, $\dfrac{(w_1 - w)(A + B)^n}{w_1(A + B)^n - wA^n}\cdot\rho$.

17. $D\left\{\frac{a}{b} + \left(1 - \frac{a}{b}\right)\left(\frac{R}{R + b}\right)^n\right\}$ 18. 94 ft.

CHAPTER XII

TENSION IN PIPES

88 Tension in the walls of a cylindrical vessel containing a gas

Consider a cylindrical vessel made of some thin flexible substance and containing gas at a certain pressure, then the intensity of the gas pressure will be uniform over the whole internal surface

Suppose a line is drawn in the surface parallel to the length of the cylinder [Fig 142 (a)] then to prevent the material of the surface from splitting along this or a parallel line the strength of the fibres of the material must provide forces F Thus in any plane circular section of the cylinder there must be circumferential forces dependent on the pressure of the enclosed gas This circumferential force per unit length of the cylinder is known as the *circumferential tension or hood tension* Let this force per unit length (inch) be T lb wt and suppose the enclosed gas is under a pressure of p lb wt per sq in, then we may derive T as follows Consider Fig 142 (b), it shows the forces acting on a unit length of the upper part of the cylinder By resolving these forces vertically we derive the hoop tension If the radius of the cylinder is r then from Chapter VII the upward force (per unit length) is p $2r$ lb wt Thus

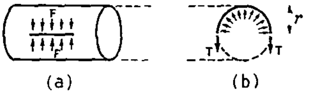


Fig 142

$$2T = 2pr,$$

and $$T = pr$$

In addition to this, the material must be capable of withstanding the tendency to split along the circumference of any circular section by supplying forces S [Fig 143 (a)]

Thus there is also a *longitudinal tension* in the surface material and if the longitudinal pull per unit length is denoted by L, we may determine its value by considering the equilibrium of a plane end [Fig 143 (b)]

The resultant thrust on a plane end due to the gas is $\pi r^2 . p$, and this is balanced by the longitudinal pull L per unit length, acting round the circumference of the circular end of length $2\pi r$, as shown.

Hence $$2\pi r \mathrm{L} = \pi r^2 p,$$

and $$\mathrm{L} = \frac{pr}{2}.$$

Thus the longitudinal tension is half the hoop tension.

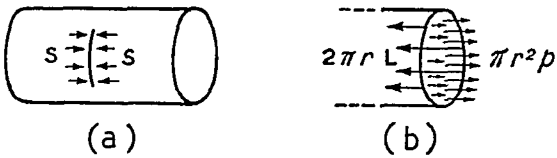


Fig. 143.

This foregoing theory assumes that there is no gas outside the cylinder, and consequently no pressure on the outside of the cylinder.

If, in fact, there is an atmospheric pressure P outside, then the resultant pressure on the internal surface of the cylinder will be $p - \mathrm{P}$, and thus $$\mathrm{T} = (p - \mathrm{P})\, r,$$

and $$\mathrm{L} = \tfrac{1}{2}(p - \mathrm{P})\, r.$$

89. Tension in thin pipes containing liquid

We restrict ourselves to the case where the pipes are horizontal and suppose that the liquids are under sufficiently large pressures that we may neglect pressure differences in the liquid due to slightly different depths. Thus we assume that the pressure over the internal surface of the pipe is uniform. This will be very nearly so if either the pressure is large or the diameter of the pipe is small or both.

The procedure, then, is exactly the same as for gases, but we may make the problem more definite by considering a pipe of small finite thickness and obtain expressions for the circumferential or hoop stress and the longitudinal stress in the material of which the pipe is composed. We have met the terms stress and tensile stress before, but we may consider them again now. For example, if a uniform metal rod of 3 sq. in. cross-section is under tension by a force of 15 ton wt., then the

$$\text{tensile stress} = \tfrac{15}{3} = 5 \text{ ton per sq. in.}$$

Consider a pipe of internal radius r and thickness t (small compared to r) [Fig. 144 (a)]. To find the circumferential or hoop stress, we consider a circular section through the pipe and consider the forces acting on half of it, exactly as we did for a cylinder containing gas. For a length l of the pipe, the area of metal on each side in contact with the lower half of the pipe is tl square units (suppose square inches). Then

if the circumferential or hoop stress in the metal is f ton per sq in, the total force to resist bursting is $2ftl$ But this is balanced by the bursting force which is $2rl\,p$ Hence

$$2ftl = 2rlp,$$

$$\text{i e} \quad f = \frac{pr}{t}$$

To determine the longitudinal stress, say f', consider Fig 144 (b) This represents the forces on the plane end of the pipe The total bursting force due to the liquid is $\pi r^2 p$ If t is small the area of

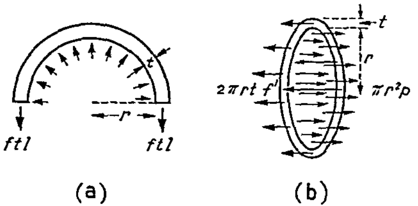


Fig 144

metal in the cross section is $2\pi r\,t$ and hence this area multiplied by f is the total resisting force

Thus $$2\pi rtf' = \pi r^2 p$$

and $$f = \tfrac{1}{2}\frac{pr}{t}$$

so that the longitudinal stress is half the hoop stress

The reason we have postulated that the pipe should be thin is that in a thin pipe the stress over the whole thickness of metal may be taken as uniform, but for a thick pipe this is no longer approximately true, the fibres of metal nearer the centre being more heavily stressed than those further away from the centre The theory of the strength of thick pipes is beyond the scope of this book, but reference may be made to books on hydraulics

Example 1 —*A wrought iron pipe, internal diameter $\frac{1}{2}$ ft, has to withstand an internal pressure of 1300 lb per sq in, the thickness of the pipe being $\frac{1}{2}$ in Find the hoop stress*

$$\text{Hoop stress} = \frac{pr}{t}$$

$$= \frac{1300 \times 3}{\frac{1}{2}}$$

$$= 7800 \text{ lb. per sq. in.}$$

Example 2.—*What thickness of pipe, 1 ft. internal diameter, will be necessary to stand an internal pressure of 100 lb. per sq. in. if the stress in the pipe must not exceed 2000 lb. per sq. in.?*

$$\text{Thickness} = \frac{pr}{f}$$

$$= \frac{100 \times 6}{2000} \text{ in.}$$

$$= 0{\cdot}3 \text{ in.}$$

90. Tension in a spherical vessel containing gas

By symmetry, the tension in the spherical surface must be constant everywhere.

Consider the forces acting on one half of the sphere. If its radius is r, and the tension round the rim of the hemisphere is T per unit length, then the total pull across the rim of the hemisphere is $T.2\pi r$. If the pressure of the gas inside the sphere is p, then the total outward pressure is $\pi r^2 . p$.

Thus $$T . 2\pi r = \pi r^2 p,$$

i.e. $$T = \tfrac{1}{2} pr,$$

which is half the hoop tension in the case of a cylinder.

Again, if we have a thin metal sphere, of thickness t, and stress f, this equation becomes

$$f . 2\pi r t = \pi r^2 p,$$

i.e. $$f = \tfrac{1}{2}\frac{pr}{t}.$$

By comparing this expression with the hoop stress in the case of a cylinder we see that if a cylindrical boiler has hemispherical ends, these ends need only be half the thickness of the cylindrical part to withstand the same pressure. This, of course, will only be true if the thickness is small compared to the radius. In practice, the weakness caused by rivets and joints must also be taken into consideration.

Exercises XII

(1) A uniform solid rod of circular cross section, diameter d in, is under a tension of P ton wt What is the intensity of tensile stress ?

(2) What fluid pressure would just burst a cylindrical metal pipe of internal diameter 1 ft, the thickness of the metal being 0 2 in, if the metal can bear a tension of 9000 lb per sq in ?

(3) Find the hoop stress in a cylindrical steam boiler of internal diameter 7 ft, if the thickness of the plates is ½ in and the steam pressure is 120 lb per sq in

(4) If, in Question 3 we take into consideration an atmospheric pressure of 14 7 lb per sq in outside the boiler, what is then the hoop stress ?

(5) A cylindrical boiler with hemispherical ends of the same internal radius contains steam at a pressure of 10P, where P is the atmospheric pressure If the hemispherical ends of thickness t_1 are exposed to the atmosphere while the cylindrical part, of thickness t_2 is exposed to an external pressure of 3P, find the ratio of t_1 to t_2 in order that the stress in the hemispherical ends shall be the same as the hoop stress in the cylindrical part

(6) A horizontal pipe with thin walls and inner radius r is full of water at pressure p If there is a bend in the pipe in the form of a quadrant of a circle show that there is a longitudinal tensile stress in the pipe near the bend, of magnitude $pr/2t$, where t denotes the thickness of the pipe (The assumption is to be made that no external horizontal forces act on the pipe near the bend)

If the bend were greater or less than 90° would this have any effect on the tension ? (Inter Eng)

(7) A *vertical* pipe, of height h ft and bore of diameter d in, contains water Where is the hoop stress a maximum, and what is this maximum value, if the thickness of the pipe is t in, and the density of water is w lb per cub ft ?

(8) A spherical balloon of radius 16 in is filled with gas at a pressure of 20 lb per sq in. and rises from the earth If the casing has a thickness of $\frac{1}{16}$ in and can just stand a stress of 800 lb per sq in find the height at which it bursts Assume that the atmospheric pressure is 14 77 lb wt per square inch at the earth s surface and that it varies at any height above the earth s surface in accordance with the expression obtained in § 78

ANSWERS

1. $\frac{4P}{\pi d^2}$ ton wt per sq in 2. 300 lb wt /sq in

3 10 080 lb wt /sq in. 4. 8845·2 lb wt /sq in 5 9 14

6 No 7 At base, $\frac{dhw}{2t}$ lb wt /sq ft 8 10,278 ft

CHAPTER XIII

WORK DONE IN HYDROSTATIC PROCESSES

91. Work

When a force is applied to a body and the point of application moves, *the force is said to do work, and the amount of work done is measured* by the product of the force and the distance through which the point of application moves *in the direction of the force.*

Thus if a force of F lb. wt. move its point of application through a distance s ft., in the direction of the force, the work done $= Fs$ ft-lb. wt. [Fig. 145 (a)]. If, as in Fig. 145 (b), the force acts at an angle θ with the direction in which the point of application is constrained to move (AB), then we require the resolved part of the force in the direction AB. This is F cos θ, so that the work done moving the point of application from A to B is $Fs \cos \theta$ ft.-lb. wt. The resolved part of this work perpendicular to AB (*i.e.* F sin θ) does no work because there is no displacement in this direction.

Now suppose that the force is variable, depending on the exact position between A and B. Consider a small element of distance δs and suppose that the force has a value F in the direction of the displacement, supposed constant throughout this small displacement. Then the work done moving the point of application through this distance δs is $F . \delta s$, hence the total work done from A to B is

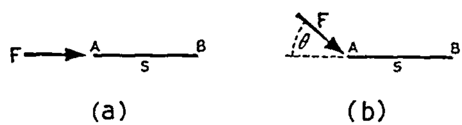


Fig. 145.

$$\int F ds,$$

where the integration is evaluated between limits which correspond to A and B. Thus, if with respect to some origin in the direction of AB, A is the point where $s = s_1$ and B is the point where $s = s_2$ (*i.e.* AB $= s_2 - s_1$), then

$$\text{work done} = \int_{s_1}^{s_2} F ds$$

and, of course, the integral cannot be evaluated until F is known as a junction of s (*i.e.* the form of variation of the force with the distance must be known).

In this chapter we shall only be concerned with work done in hydrostatic processes such as the transference of liquids from one level to another, the work done in altering the position of a floating body relative to the liquid, and the work done in the expansion of gases.

Much help may be obtained, especially in the case of variable forces, by drawing a *force space diagram* or *work diagram*. It consists simply of plotting the force against the distance, and, from the relations above, the work done will be represented by the area under the curve so obtained. Consider Fig 146 (*a*). Here we have a constant force F plotted vertically against distance moved (plotted horizontally). The distance moved is from s_1 to s_2 so that the shaded area represents the work done, i e F $(s_2 - s_1)$. In Fig 146 (*b*) we have a variable force, and the shaded area here is $\int_{s_1}^{s_2} Fds$, which is equal to the work done

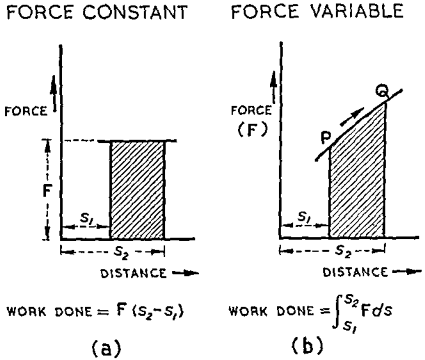


Fig 146

Before proceeding with the solution of examples it is convenient to notice here an application of the force space diagram which is of particular importance to engineers. Notice, first of all that in Fig 146 (*b*) we have supposed the curve traced out from P to Q, i e that s was increasing. If it were traced in the other direction, i e with s decreasing, the shaded area would give the work done *against* the force F (by other forces), which would be negative work as far as the force F is concerned.

Suppose we have a variable force which moves its point of application from A to B [Fig 147 (*a*)], by the path PQR. Then the area PQRBA

represents the positive or useful work done by the force. If the point of application, when it reaches B, is moved back to A (by other forces) by the path RSP, then this path traces out the magnitude of the variable force as work is done against it. This negative work is represented by the area BRSPA, and hence the total positive or useful work done by the variable force as its point of application moves from A to B and back is represented by the difference between the areas PQRBA and BRSPA,

i.e. useful work done = shaded area PQRS.

This theory is used in the determination of the amount of work done by a machine operated by fluid pressure on a piston. For instance, in a steam engine the forward stroke is obtained by the pressure of steam pushing the piston out and driving the machinery. Most of the steam then escapes and the piston is then returned to its original

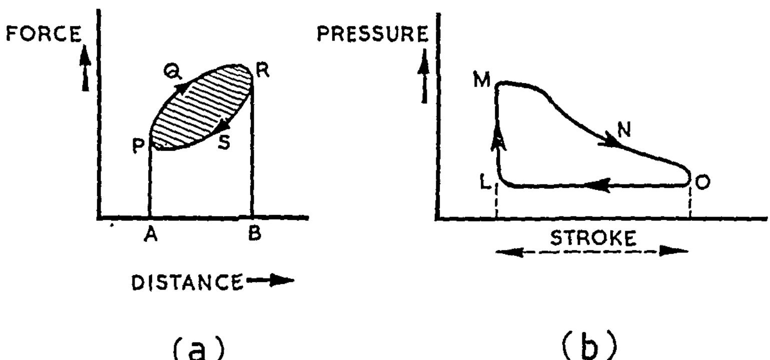


Fig. 147.

position, against the diminished steam pressure, by the momentum of the machinery. More steam enters and the process is repeated. An instrument, called the indicator, is attached to the machine and automatically draws the work diagram for the machine. It is then called the *indicator diagram* and is often of the form shown in Fig. 147 (*b*). LM represents the rise in pressure as steam is admitted into the cylinder. MNO indicates the fall in steam pressure as the piston is driven forwards throughout its stroke and OL indicates the steam pressure against which the piston is returned to its original position. These diagrams are important to the engineer since they give the total work done by a particular engine, and also the variation of pressure from point to point as the piston traverses its cylinder.

It must be realised that although the engineer uses units of foot pound weight for work done, equally well we may use the absolute unit of foot-poundal.

We proceed to a few examples.

obtained by drawing the line $F = xa\rho$, where F is the resultant upthrust (Fig 149) This is a straight line of the form $y = mx$, and consequently the work done is the area of the triangle OAB, where $OA = h - l$ Since the height of an ordinate at a distance x is given by $xa\rho$ the length AB is equal to $(h - l)\,a\rho$, hence

$$\begin{aligned} \text{total work done} &= \text{area of triangle OAB} \\ &= \tfrac{1}{2}\text{OA AB} \\ &= \tfrac{1}{2}(h - l)\ (h - l)\,a\rho \\ &= \tfrac{1}{2}a\rho\,(h - l)^2, \end{aligned}$$

which is the same as that obtained in equation (iii),

$$= \tfrac{1}{2}\frac{ah^2}{\rho}(\rho - \sigma)^2 \text{ ft lb wt as before}$$

92 Work and potential energy

The Potential Energy of a body is its capacity to do work in virtue of its position For instance, if a mass of 2 lb is held at a height of 7 ft above the floor it possesses a potential energy of 14 ft lb wt with respect to the floor *i e* it can do 14 ft lb wt of work in dropping to the floor

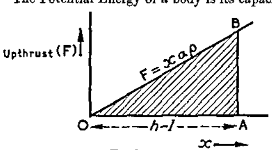


Fig 149

Thus in hydrostatic processes if work is done on a body against gravity, there is a corresponding increase in the potential energy of the body Again if gravity is allowed to do work on a body there is a corresponding decrease in its potential energy Consequently we may frequently solve problems on the work done in a certain process by considering the changes occurring in the potential energies of the bodies affected by the process A body of liquid will possess a potential energy and if the configuration of the body of liquid is changed its change in potential energy will be measured by the product of its weight and the vertical height through which its centre of gravity is moved This is exemplified in Example 1 below

Example 1 —*The bottom of a cylindrical well of cross section a sq ft, is h ft below ground level the height of water in the well being d ft Find the work done in pumping all the water up to ground level and the corresponding change in potential energy*

Consider the water contained in the well between horizontal planes at depths x and $x + \delta x$ below ground level (Fig 150)

The volume of the element $= a\delta x$ cub. ft.;

$\therefore$ the weight of the element $= wa\delta x$ lb. wt.,

where w is the weight per cubic foot of water.

Therefore, the work done in bringing this weight of water to ground level $= x . wa\delta x$ ft.-lb. wt.;

$\therefore$ total work done in bringing all water to ground level

$$= \int waxdx,$$

where x ranges from $h - d$ to h,

i.e. total work done

$$= \int_{h-d}^{h} waxdx$$

$$= wa\left[\frac{x^2}{2}\right]_{h-d}^{h}$$

$$= \tfrac{1}{2}wa\,[h^2 - (h-d)^2]$$

$$= \tfrac{1}{2}wa\,(2hd - d^2)$$

$$= awd\left(h - \frac{d}{2}\right) \text{ ft.-lb. wt.}$$

But $awd =$ total weight of water,

and $h - \dfrac{d}{2} =$ depth of C.G. of water below ground level.

Therefore, with reference to the bottom of the well, the water has gained a potential energy of

$$awd\left(h - \frac{d}{2}\right) \text{ ft.-lb. wt.}$$

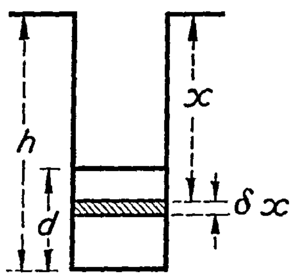


Fig. 150.

Example 2.—*A cold water tank is in the form of a cube of edge 4 ft. and is full of water. It is connected with a cylindrical hot water tank of the same cubical content, but of height 6 ft., standing with its axis vertical, so that the upper end of the hot water tank is 10 ft. below the base of the cold water tank. If the water flows under gravity from the upper tank to the lower, find the loss in potential energy.*

The volume of water in the upper tank $= 4 \times 4 \times 4$ cub. ft.;

$\therefore$ weight of water in upper tank $= 4 \times 4 \times 4 \times 62\tfrac{1}{2}$ lb. wt.

$= 4000$ lb. wt.

The C.G. of the upper tank is $2 + 10 + 3$ ft. above the C.G. of the lower tank;

$\therefore$ loss of potential energy $= 4000 \times 15$ ft.-lb. wt.

$= 60{,}000$ ft.-lb. wt.

Example 3 —*Solve Example 2 § 91, by considering the change in potential energy of the system*

Turn again to Fig 148

When the block is pressed down, it loses potential energy, but there is a gain of potential energy by the liquid since the liquid which is displaced by depressing the cylinder rises to the surface

In the final position there is an extra volume of liquid displaced of amount $(h - l)\,a$ cub ft, i e its weight is $(h - l)\,a\rho$ lb wt

Its C G was originally at a depth $l + \dfrac{h-l}{2}$ below the surface

Hence,

$$\text{potential energy gained by liquid} = (h - l)\,a\rho\left(l + \frac{h-l}{2}\right)$$

$$= \tfrac{1}{2}a\rho\,(h - l)(h + l) \text{ ft lb wt}$$

$$= \tfrac{1}{2}a\rho\,(h^2 - l^2) \text{ ft lb wt} \qquad \text{(i)}$$

The weight of cylinder $= ha\sigma$ lb wt,

and its C G is moved downwards a distance $h - l$ ft,

potential energy lost by cylinder

$$= ha\sigma\,(h - l) \text{ ft lb wt}$$

$$= la\rho\,(h - l) \text{ ft lb wt} \qquad \text{(ii)}$$

since $\qquad ha\sigma = la\rho,$

total potential energy gained by system

$$= \text{(i)} - \text{(ii)}$$

$$= \tfrac{1}{2}a\rho\,(h^2 - l^2) - la\rho\,(h - l)$$

$$= \tfrac{1}{2}a\rho\,(h - l)(h + l - 2l)$$

$$= \tfrac{1}{2}a\rho\,(h - l)^2 \text{ ft lb wt}$$

which is the same expression as that obtained in Example 2 § 91, and equals $\tfrac{1}{2}\dfrac{ah^2}{\rho}(\rho - \sigma)^2$ ft lb wt

This is the total potential energy gained by the system and consequently is equal to the work done in depressing the cylinder to the position required

Example 4 —*A sphere of weight W and radius a is floating half immersed in a large vessel of water Find the work required to raise it just clear of the water*

Here, potential energy is gained by the sphere, since its C G is raised a distance a vertically upwards, but potential energy is lost by the water which has to run down to fill up the hemispherical space left by the sphere being raised

Potential energy gained by sphere $= Wa$

The weight of water displaced by the sphere when floating is W, hence this is the weight of water which must be lowered from the surface. The C.G. of a hemisphere is at a distance of $\frac{3}{8}a$ from the centre and therefore,

$$\text{potential energy lost by water} = W.\tfrac{3}{8}a.$$

Therefore,

$$\begin{aligned}\text{total gain of potential energy by system}\\ &= Wa - \tfrac{3}{8}Wa\\ &= \tfrac{5}{8}Wa,\end{aligned}$$

and this is, therefore, the amount of work which is required to raise the sphere just clear of the water.

Example 5.—*A cylinder of cross-section a floats with its axis vertical in water contained in a cylindrical vessel of cross-section 6a. If the density of the water is ρ, and the cylinder is lifted through a height x (but not right out of the water), show that the increase in the potential energy of the system is* $\frac{3}{5}a\rho x^2$.

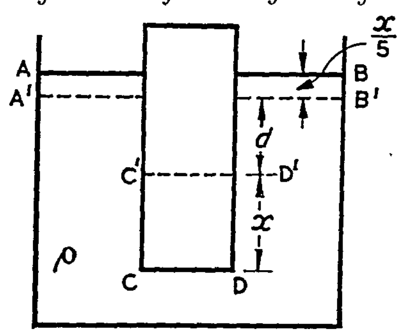


Fig. 151.

Method (i).—By finding work done directly.

Consider Fig. 151. AB represents the original level of the water and CD the base of the cylinder. When the cylinder is raised a distance x so that its base is at C′D′, the level of the water has dropped to A′B′, so that the only length then immersed is the difference in height of A′B′ and C′D′. Let this distance be d.

When the cylinder is raised a distance x, the volume of water filling the space it occupied between CD and C′D′ is ax; and this volume has come from the lowering of the level from A to A′.

$$\begin{aligned}\text{The area of the water surface} &= 6a - a\\ &= 5a;\end{aligned}$$

$$\therefore \quad \text{volume of water between AB and A'B'} = 5a \times AA';$$

$$\therefore \quad 5a \times AA' = ax;$$

$$\therefore \quad AA' = \tfrac{1}{5}x.$$

Let weight of cylinder = W.

By Archimedes' Principle this equals the weight of water displaced when in equilibrium,

i.e. $$W = a\rho\left(x + d + \frac{x}{5}\right) \quad \text{(i)}$$

After the cylinder is raised a distance x, the length immersed is d

weight of water displaced $= ad\rho$, and this is the upthrust

Therefore,

$$\text{resultant force downwards} = \text{weight of cylinder} - \text{upthrust}$$

$$= W - ad\rho$$

$$= a\rho\left(\frac{6x}{5} + d\right) - ad\rho \text{ from (i)}$$

$$= \frac{6xa\rho}{5} \qquad \text{(ii)}$$

This is the resultant force downwards for a displacement x. Let the displacement of a position intermediate between CD and C'D be z, then resultant force downwards for a displacement z

$$= \frac{6za\rho}{5}$$

Therefore, the work done in increasing this displacement by δz is

$$\frac{6za\rho}{5}\,\delta z,$$

total work done for displacement x

$$= \int_0^x \frac{6za\rho}{5}dz$$

$$= \frac{6a\rho}{5}\int_0^x z dz$$

$$= \frac{6a\rho}{5}\,\frac{x^2}{2}$$

$$= \tfrac{3}{5}a\rho x^2,$$

and this is equal to the increase in the potential energy of the system

Method (ii) —By considering the change of potential energy

Using the same notation as before, and considering the change in potential energy for a final displacement x,

$$\text{potential energy gained by cylinder} = Wx \qquad \text{(iii)}$$

The water which is lowered has a weight $ax\rho$, and its C G, originally at a depth half way between B and B, is lowered to mid way between D and D, i e a distance of $\frac{x}{10} + d + \frac{x}{2}$,

$$\text{potential energy lost by water} = ax\rho\left(\frac{x}{10} + d + \frac{x}{2}\right)$$

$$= ax\rho\left(\frac{3x}{5} + d\right) \qquad \text{(iv)}$$

$\therefore$ total potential energy gained by system

$$= \text{(iii)} - \text{(iv)}$$
$$= \mathrm{W}x - ax\rho\left(\frac{3x}{5} + d\right)$$
$$= a\rho x\left(x + d + \frac{x}{5}\right) - ax\rho\left(\frac{3x}{5} + d\right) \text{ from (i)}$$
$$= a\rho x\left(\frac{6x}{5} + d - \frac{3x}{5} - d\right)$$
$$= \frac{3}{5}a\rho x^2, \text{ as before.}$$

Method (iii).—By finding the area of the work diagram.

In Method (i) we determined the resultant force downwards when the cylinder was raised a distance x; it was [equation (ii)]

$$\text{resultant force downwards} = \frac{6xa\rho}{5}.$$

This gives a straight line if the force is plotted against x, so that the work diagram is a triangle exactly as in Fig. 149. Referring to that figure OA is now x, AB is $\frac{6xa\rho}{5}$, hence

$$\text{area of work diagram} = \tfrac{1}{2}\mathrm{OA}.\mathrm{AB}$$
$$= \frac{1}{2}x.\frac{6xa\rho}{5}$$
$$= \tfrac{3}{5}a\rho x^2,$$

and this is equal to the increase in potential energy.

93. Expansion of gases

In Chapter X, § 75, we obtained the fundamental relation for a gas [equation (xii)] in the form

$$pv = \mathrm{RT} \quad \text{.......................(i)}$$

showing that the product of the pressure and volume of a given mass of gas is proportional to its absolute temperature.

When dealing with the expansion of a gas there are two important cases to consider; if the temperature remains constant during the expansion, this is then said to be an *isothermal* expansion. Such an expansion must take place very slowly, so that no heat is absorbed. Consequently, from the fundamental relation above (i), the relation between pressure and volume for isothermal expansion of a gas will be given by

$$pv = \text{constant} \quad \text{...................(ii)}$$

since the temperature is constant.

therefore, from equation (ii), the work done in the expansion from a volume v_0 to a volume v_1 is

$$\begin{aligned} \text{work done} &= \int_{v_0}^{v_1} p\,dv \\ &= \int_{v_0}^{v_1} \frac{C}{v}\,dv \\ &= C\Big[\log v\Big]_{v_0}^{v_1} \\ &= C \log \frac{v_1}{v_0} \\ &= p_0 v_0 \log \frac{v_1}{v_0}, \end{aligned}$$

where p_0 is the original pressure. Similarly, the work done in the contraction of a gas from a volume v_0 to v_1

$$= p_0 v_0 \log \frac{v_0}{v_1}.$$

(ii) *Work done during an adiabatic expansion.*

The equation for adiabatic change is

$$pv^\gamma = K \text{ (a constant)}.$$

Therefore

$$\begin{aligned} \text{work done} &= \int_{v_0}^{v_1} p\,dv \\ &= \int_{v_0}^{v_1} \frac{K}{v^\gamma}\,dv \\ &= K\left[\frac{1}{1-\gamma} v^{1-\gamma}\right]_{v_0}^{v_1} \\ &= \frac{K}{1-\gamma}\left(v_1^{1-\gamma} - v_0^{1-\gamma}\right) \\ &= \frac{1}{1-\gamma}\left(p_1 v_1^\gamma v_1^{1-\gamma} - p_0 v_0^\gamma v_0^{1-\gamma}\right) \end{aligned}$$

since $p_0 v_0^\gamma = K = p_1 v_1^\gamma$,

$$\begin{aligned} &= \frac{1}{1-\gamma}(p_1 v_1 - p_0 v_0) \\ &= \frac{R}{1-\gamma}(T_1 - T_0) \end{aligned}$$

since $p_0 v_0 = RT_0$ and $p_1 v_1 = RT_1$.

We proceed to solve some examples.

Example 1.—*One cubic foot of air, originally at a pressure of* 100 *lb. per sq. in., expands to* 10 *cub. ft. Find the work done, assuming* (a) *isothermal expansion, and* (b) *adiabatic expansion.*

(*a*) Isothermal expansion.

The equation is $pv = C$;

$$\therefore \quad C = 100 \times 144 \text{ lb./sq. ft.} \times 1 \text{ cub. ft.}$$
$$= 100 \times 144 \text{ ft.-lb.}$$

$$\text{The work done} = \int_1^{10} pdv$$
$$= C\int_1^{10}\frac{dv}{v}$$
$$= C \log_e \tfrac{10}{1}$$
$$= 100 \times 144 \times 2{\cdot}30258 \text{ ft.-lb. wt.}$$
$$= 37{,}557{\cdot}4 \text{ ft.-lb. wt.}$$

(*b*) Adiabatic expansion.

The equation is $pv^{\gamma} = K$;

$$\therefore \quad K = 100 \times 144 \times 1^{\gamma}$$
$$= 14{,}400.$$

$$\text{The work done} = \int_1^{10} pdv$$
$$= \int_1^{10}\frac{K}{v^{\gamma}}\,dv$$
$$= \frac{K}{1-\gamma}\left(10^{1-\gamma} - 1^{1-\gamma}\right)$$
$$= \frac{14{,}400}{-0{\cdot}4}\left(10^{-0{\cdot}4} - 1\right)$$
$$= 21{,}600 \text{ ft.-lb. wt.}$$

Example 2.—*A horizontal circular cylinder, of cross-section* S *and length* a, *is full of air at atmospheric pressure* P. *The air is slowly compressed by an air-tight piston until it only fills a length* b *of the cylinder. Assuming the temperature remains constant, find the work done during the compression.*

Consider the force acting on the gas in the cylinder when it has been compressed to a length x (Fig. 152). Let the pressure of the gas be p, then since atmospheric pressure is acting on the outside of the piston, the total pressure resisting compression is $p - P$.

Therefore, the total force resisting compression $= (p - \mathrm{P})\mathrm{S}$

Thus, the work done in a negative displacement (i e $-\delta x$)

$$= -(p - \mathrm{P})\,\mathrm{S}\ \delta x,$$

Total work done in compression

$$= -\int_a^b (p - \mathrm{P})\,\mathrm{S}\ dx \qquad \text{(i)}$$

since the length of cylinder decreases from a to b

But, since the compression is isothermal

$$pv = \text{constant},$$

$$\text{i e} \quad \mathrm{P\ S}a = p\ \mathrm{S}x,$$

$$p = \frac{\mathrm{P}a}{x}$$

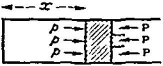


Fig 152

Thus (i) becomes

$$\text{Total work done} = -\int_a^b \left(\frac{\mathrm{P}a}{x} - \mathrm{P}\right)\mathrm{S}\ dx$$

$$= \mathrm{PS}\int_a^b \left(1 - \frac{a}{x}\right) dx$$

$$= \mathrm{PS}\,(b - a) - \mathrm{PS}a \log\frac{b}{a}$$

$$= \mathrm{PS}\,(b - a) - \mathrm{PS}a \log\frac{a}{b}$$

Exercises XIII

(1) Find the work done in pumping 1000 gall. of water through a vertical height of 24 ft (1 gall. of water weighs 10 lb)

(2) A cylindrical well has a cross-sectional area of 10 sq ft , and the bottom of the well is 30 ft below ground. If there is 10 ft of water in the well, find the work done in bringing it all up to ground level.

(3) The cross-sectional area of a cylindrical steam engine is 1 sq ft and the length of the stroke is 2 ft If the mean steam pressure during a stroke is 50 lb per sq in., find the work done per stroke

(4) The direction of a force F is constant and coincides with that of the displacement of the particle on which it acts. A B C are three points on the line of the displacement such that AB = BC = 5 ft. When P lies between A and B, $\mathrm{F} = 20 \times \mathrm{AP}$ lb., and when P lies between B and C, $\mathrm{F} = 2500\ \mathrm{AP}^2$ lb Draw the work diagram and by estimation from the graph find the total work done from A to C Check your result by calculation (Inter Eng)

(5) A solid cube floats with two faces horizontal in a large tank of water. The specific gravity of the cube is 0·75 and its volume is 1 cub. ft. Find the work done in raising the cube just clear of the water.

(6) A cylinder is floating with its axis vertical in a large tank of water. If it is pushed down so that an additional length x of its axis is immersed, show that the work done on the cylinder is $\frac{1}{2}wx$, where w is the additional weight of the water displaced.

(7) A sphere of weight W lb. and radius a ft. rests at the bottom of a cylindrical bucket, of radius b ft., which is filled to a depth h ft. with water, h being greater than $2a$. Prove that the work which must be done to lift the sphere just clear of the water is

$$W\left(h - \frac{4a^3}{3b^2}\right) - S\left(h - a - \frac{2a^3}{3b^2}\right) \text{ ft.-lb. wt.}$$

where S is the weight of water displaced by the sphere.

(8) A cylindrical block of wood, whose cross-sectional area is A, is floating in a cylinder (cross-sectional area B) containing water. Show that the work done in pushing the block down to any position where it is not wholly immersed is

$$\tfrac{1}{2}Pl\left(\frac{B}{A} - 1\right)$$

where l is the change in water level and P is force applied to the block in its final position. Solve the problem:—

(i) by considering the changes in potential energy, and

(ii) by integration.

From the expression obtained for the resultant upthrust in any position, draw the work diagram and hence evaluate the total work done as an area.

(9) If, in the last question, the weight of the block is W and the weight per unit volume of water is w, show that the work done in lifting the block just clear of the water is

$$\frac{W^2}{2ABw}(B - A).$$

(10) A sphere of weight W and radius a is floating half-immersed in a large vessel of water. Find, by integration, the work required to raise it just clear of the water and compare the result with Example 4, § 92.

(11) Air at 87° C. expands adiabatically so that its pressure is halved. Find the resulting temperature.

(12) A gas has a volume of 100 cub. ft. at a pressure of 10 lb. per sq. in. and is compressed to 20 cub. ft. Find the resulting pressures and the work done if

(i) the compression is isothermal, and

(ii) the compression is adiabatic.

(13) A gas contained in a flexible vessel expands adiabatically from a volume v_0 to a volume v_1. If the pressure of the gas is originally p_0 and there

is an external atmospheric pressure of P during the expansion show that the work done is

$$\frac{1}{\gamma - 1}(p_0v_0 - p_1v_1) - P(v_1 - v_0)$$

where p_1 is the final pressure of the gas

(14) Gas expands adiabatically from pressure p_0 and volume v_0 to volume $v_0(1 + a)$ and then contracts isothermally to volume v_0. Show that the work done by the gas is

$$p_0v_0\left[\frac{1}{\gamma - 1} - (1 + a)^{1-\gamma}\left\{\frac{1}{\gamma - 1} - \log(1 + a)\right\}\right]$$

Show that if a^3 may be neglected this reduces to $\frac{1}{2}p_0v_0a^2(\gamma - 1)$

(15) A sphere of radius a is held immersed at the bottom of a hollow cylindrical cylinder, internal radius $2a$ which is filled to a depth $3a$ with water. The sphere is released and takes up its position of equilibrium. If the specific gravity of the sphere is $\frac{1}{2}$ and its weight is W, show that the loss of potential energy is $\frac{37Wa}{24}$

ANSWERS

1 240 000 ft lb wt 2 155 750 ft lb wt 3 14 400 ft lb wt

4 500 ft lb wt 5 17 52 ft lb wt 10 $\frac{5Wa}{8}$ 11 2.3° C

12 (i) 50 lb wt per sq in 231 759 4 ft lb wt

(ii) 95 19 lb wt per sq in 325 368 0 ft lb wt

MISCELLANEOUS EXERCISES

(1) The plunger of a Bramah press has a diameter of 1 in. and is operated by a lever whose velocity ratio is 12 : 1. The diameter of the ram is 10 in., and the efficiency of the press is 85 per cent. Find the force exerted by the ram when a force of 35 lb. wt. is applied to the handle of the lever. (Inter. Sc.)

(2) Define *specific gravity*.

The apparent weights of a piece of glass when completely immersed in water and glycerine (of specific gravity 1·25) are 60 grm. and 50 grm. respectively. Determine the specific gravity of the glass. (Inter. Sc.)

(3) A canal lock-gate is 12 ft. broad and the depths of water at the opposite sides of the gate are 16 and 10 ft.; find in tons weight the magnitude of the resultant water thrust on the gate, assuming that a cubic foot of water has a mass of 62·5 lb.

(4) A cube, each side of which is 6 in. long, lies at the bottom of a water tank 5 ft. deep. Calculate in pounds weight the resultant fluid pressure on the upper horizontal face of the cube, and also on a vertical face, assuming that the atmospheric pressure is that due to 34 ft. of water, and that a cubic foot of water weighs $62\frac{1}{2}$ lb.

(5) Two circles are drawn on a vertical wall of a reservoir; they touch each other externally, and the centre of the one (radius r) is vertically below the centre of the other (radius R). The water in the reservoir rises so that the upper circle is just immersed. If $x = R/r$, show that, when the resultant pressures on the two circles are equal, x is given by the equation

$$x^3 - 2x - 1 = 0.$$

Show that $x = -1$ is a solution of this equation and hence that the only positive solution is

$$x = \tfrac{1}{2}(1 + \sqrt{5}).$$

(6) Show that the magnitude of the thrust on a plane surface of area A immersed in a liquid is ρAd, where ρ is the density of the liquid and d is the depth of the mean centre of the area below the free surface.

A hollow regular pyramid, made of thin sheet metal, has a square base of side $2a$ and vertical height $3a$. The pyramid is completely filled with water, the weight of the water being four times that of the pyramid itself. If the whole rests with the base of the pyramid on a horizontal plane, show that the thrust of the water on the base of the pyramid is 2·4 times the thrust of the whole on the plane.

Find the magnitude and direction of the thrust of the water on a slant face of the pyramid and use these results to explain the apparent anomaly above. (H.S.C., I.)

(7) A heavy uniform bar of specific gravity 0 75 partly rests in water, the upper end being fixed at a point which is a quarter of its own length above the surface Show that a position of rest is at an inclination of 30° to the horizontal

(8) A hollow cone consisting of a curved surface closed by a circular base both made of thin sheet metal, is 12 in high and has a radius of 5 in The cone is to rest completely submerged in water with its vertex fixed and its axis horizontal, find the necessary weight of metal per square inch of surface, assuming that a cubic foot of water has a mass of 1000 oz

(9) Prove that a diver who is able to stand a pressure of n atmospheres can descend to a depth of n times the height of the water barometer, and that if the diving dress when inflated displaces V cubic feet, it must be loaded with 64 V lb of lead in sea water of specific gravity 1 024 for the diver to walk about the bottom of the sea with the same ease as on dry land

(10) A rectangular tank is divided into two compartments by a vertical diaphragm The two compartments are then filled to heights h, k with liquids of densities ρ, σ respectively Show how to choose h and k so that the resultant of the pressures on the diaphragm shall reduce to a couple, and find the magnitude of this couple per unit breadth of the tank

(11) A cylinder of radius r floats in liquid of density σ inside a cylindrical vessel of radius a Show that if a mass W be placed on the floating cylinder, it will sink by an amount

$$\frac{W}{\pi\sigma}\left(\frac{1}{r^2}-\frac{1}{a^2}\right)$$

(12) A thin hemisphere without base is placed in close contact with a metal disc which is horizontal and lowermost if the common radius of the hemisphere and disc be 1 in and the weight of the disc 4 oz, find the extent to which the enclosed air must be rarefied in order that a weight of 25 lb may be suspended from the centre of the disc (Take the atmospheric pressure as 15 lb per square inch)

(13) State and prove the principle of Archimedes

A piece of metal which weighs 10 kilog floats in mercury with 5/9ths of its volume immersed Find the volume and density of the metal, assuming the specific gravity of the mercury to be 13 5 (Inter Sc)

(14) A cube of ice, specific gravity 0 918, floats in sea water, specific gravity 1 026, with two faces horizontal and projects 1 cm above the surface of the water Find to what height it will project if transferred to float in fresh water (Inter Sc)

(15) A closed cubical box of side 2 ft is half full of oil, specific gravity 1 3, and half full of water It is tilted about one edge which is horizontal until the faces about this edge are inclined at 45° to the horizontal Find the liquid thrust on one of the vertical faces of the box (i) if the oil and water are not mixed, (ii) if the oil and water are thoroughly mixed Take the density of water to be 62 5 lb /ft ^{3} (Inter Sc)

(16) Prove that the depth of the centre of liquid pressure on a triangular lamina vertically immersed with one edge in the free surface is half the depth of the lowest point of the lamina.

The triangular lamina ABC is vertically immersed with the edge AB in the free surface and C at depth h. If D is the mid-point of BC, show that the depth of the centre of liquid pressure on the triangular portion ADC is $7h/12$, and find the ratio of the thrusts on ADC and ADB. (Inter. Sc.)

(17) ABC is a triangular lamina, the angle A being a right angle and the angle B 30°. It can turn freely in its own plane which is vertical, about A which is fixed in the surface of water. If it is in equilibrium with C immersed and AC making an angle of 60° with the horizontal, show that the specific gravity of the material composing the lamina is $\frac{3}{8}$. (Inter. Sc.)

(18) The cross-section of a barometer tube is one square inch, the Torricellian Vacuum is 4 in. in length, and the barometric height is 30 in.; one-fifth of a cubic inch of the external air is now introduced into the vacuum, and at the same time the absolute temperature is diminished by one-sixth; putting the contraction of the mercury out of the question, find the pressure in the vacuum.

(19) A hollow right triangular prism is closed at the top by a heavy lid ABC hinged along BC. The prism is filled with water and the whole tilted about BC, which is kept horizontal, until water just begins to escape. Show that the angle made with the horizontal by the plane ABC is the same whether A is above or below BC. (H.S.C., III.)

(20) A right circular solid cone (specific gravity >1) is suspended by a string attached to a point on the circumference of its base and rests completely immersed in water, with the point of attachment of the string in the surface of the water. If the height is double the radius of the base, show that the horizontal and vertical components of the resultant fluid pressure on its curved surface are respectively $6W/5$ and $8W/5$, where W is the weight of the water displaced by the cone.

(21) A thin hollow vessel in the shape of a paraboloid of revolution floats in a liquid of density ρ with its axis vertical and vertex downwards. If the weight of the vessel itself can be neglected, find to what height it must be filled with liquid of density σ $(\sigma > \rho)$ in order that its vertex may be at a distance h below the free surface of the first liquid.

(22) Find the centre of liquid pressure of a rectangle immersed in a liquid with one edge in the surface.

A hollow cube of side a with one face horizontal has its lower half filled with water, whilst the upper half is filled with a liquid of specific gravity 0.9. If the two liquids do not mix, show that the depth of the centre of liquid pressure on a vertical face of the cube is $149a/222$. (Inter. Sc.)

(23) A quadrilateral ABCD, with sides AB = AD, CB = CD is totally immersed vertically in a liquid with the vertex A on the surface and the diagonal BD horizontal If the perpendiculars from A and C on BD are of lengths a and b respectively, prove that the centre of pressure of the quadrilateral will be at the mid point of BD provided

$$2a = (\sqrt{5} + 1)\, b \qquad \text{(H S C, I)}$$

(24) Find the depth of the centre of pressure of a triangle immersed vertically in liquid with one side in the free surface

A triangle ABC, right angled at C, is immersed vertically in water with AB in the free surface It is then rotated about A until AC is vertical Show that the depths of the centre of pressure on the triangle ABC in the two positions are in the ratio $2 \sin \alpha \quad 3$, where α is the angle BAC (Inter Sc)

(25) Find the depth of the centre of pressure of a triangular area ABC immersed vertically in liquid with the side BC in the free surface at which the pressure vanishes, given that the depth of A below BC is d

If X is the point of the side AC such that the liquid thrusts on the triangles BCX, ABX are equal, show that the depth of the centre of pressure of the triangle ABX is

$$d\,(1 - \tfrac{1}{4}\sqrt{2}) \qquad \text{(H S C, I)}$$

(26) A closed vessel of thin sheet metal consists of a right circular cylinder, radius a, height a, closed at one end and having the rim of the other end soldered to the rim of the base of a circular cone of radius a and height a It is held with its axis vertical and conical part uppermost, and filled with water through a small orifice at the vertex of the cone If the orifice is now closed and the vessel inverted, find the ratio of the liquid thrusts on the curved conical surface in the two positions

(Inter Sc)

ANSWERS

1. 35,700 lb. wt. 2. 2·5.

3. 26·1. 4. 601·6 lb. wt., 605·5 lb. wt.

6. $2a^3w\sqrt{10}$: $\tan^{-1}(3)$ with vertical. 8. 0·635 oz./sq. in.

10. Pressure reduces to a couple if $\rho h^2 = \sigma k^2$. Magnitude of couple is $\sigma k^2 (h \sim k)/6$.

12. 46·4%. 13. $1\frac{1}{3}$ litres; 7·5 grm./cm.^{3}. 14. 7·79 mm.

15. (i) 371·2 lb. wt.; (ii) 406·8 lb. wt. 16. 3 : 1.

18. That due to 1 in. of mercury. 21. $h\sqrt{(\rho/\sigma)}$. 26. 1 : 2

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(2) The horizontal component of the reaction (R_H) exerted by the curved lamina ABCD

Hence these are equal and opposite, and since the thrust exerted by the fluid on the lamina is equal and opposite to the reaction exerted by the lamina on the fluid we have —

Resultant horizontal thrust of the fluid on the area in a given horizontal direction is equal to the thrust which would be exerted on the projection of the area on to a vertical plane perpendicular to the given direction and acts through the centre of pressure of that projected area

Example —*A hollow hemispherical shell of radius r is immersed in liquid of density ρ with a diameter vertical If the centre is at a depth h, find the resultant horizontal thrust on the hemisphere*

The projection of the hemispherical shell is a circle, radius r and centre at depth h Hence, by the theorem above, the resultant horizontal thrust on either side of the curved surface of the hemispherical shell is equal to the thrust on the circle This, from § 27 (Case 7) is $\pi r^2 \rho h$ and the depth of the centre of pressure of the circle is $h + \frac{r^2}{4h}$ from § 42 (Example 2) Thus we know the magnitude and position of the resultant horizontal thrust

51 Examples

We now propose to solve some problems concerning thrusts on curved surfaces in which we may compound the resultant vertical and horizontal thrusts to obtain a single resultant

Example 1 —*A closed cylinder of radius r ft and length l ft is just full of liquid of density ρ lb per cub ft If the cylinder is held with its axis horizontal, find the liquid thrust on each half of the curved surface determined by a horizontal plane through the axis*

The resultant vertical thrust on the upper half of the curved surface of the cylinder will be the weight of the liquid which would stand on this part if the liquid were outside instead of inside, the levels in the two cases being the same Fig 88 shows the section

Thus the volume of liquid standing on the upper half

$$= 2r^2l - \tfrac{1}{2}\pi r^2 l \text{ cub ft}$$

$$= lr^2(2 - \tfrac{1}{2}\pi) \text{ cub ft},$$

$$\text{weight of liquid} = lr^2\rho(2 - \tfrac{1}{2}\pi) \text{ lb wt}$$

Hence the resultant vertical thrust on the upper part is $lr^2\rho(2 - \frac{1}{2}\pi)$ and acts upwards as shown in Fig 88 (since it acts through the centre

of gravity of the liquid). If this upper half were divided by a vertical plane through the axis, it is clear that the horizontal thrust on one half would be equal and opposite to the horizontal thrust on the other half so that there is no resultant horizontal thrust. Consequently, the resultant vertical thrust obtained above is in fact the total liquid thrust on the upper half.

The resultant vertical thrust on the lower half is obtained in exactly the same way by finding the weight of liquid which would stand on the lower half. The section is as in Fig. 89, so that this weight

$$= \rho\,(2r^2l + \tfrac{1}{2}\pi r^2 l)$$

$$= lr^2\rho\,(2 + \tfrac{1}{2}\pi) \text{ lb. wt.}$$

As in the case above, there is no horizontal thrust, so that this represents the total liquid thrust on the lower half.

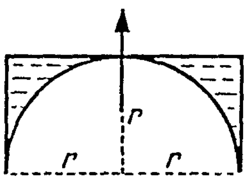

Fig. 88.

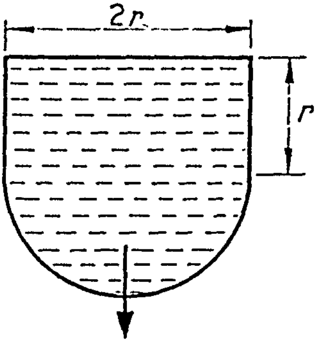

Fig. 89.

Notice that the difference between the thrusts on the lower and upper halves gives, as it must do, the weight of the cylinder of liquid.

Thus $$lr^2\rho\,(2 + \tfrac{1}{2}\pi) - lr^2\rho\,(2 - \tfrac{1}{2}\pi) = \pi r^2 l\rho.$$

Example 2.—*A basin is in the form of the part of a hemisphere of radius 5 in. between its plane face and a parallel section of radius 3 in. which forms the base of the basin. If it is placed on a horizontal table and contains water to a depth of 1 in., find the resultant liquid thrust on the curved surface, taking the weight of 1 cub. ft. of water to be $62\frac{1}{2}$ lb. [The volume of a segment of height h cut from a sphere of radius a is $\pi h^2 (a - \frac{1}{3}h)$.]* *(Inter. Sc.)*

Let ABCD (Fig. 90) represent the basin, EF the surface of the water, and O, L, M the mid-points of AD, FE, BC respectively.

Since OC = 5 in., MC = 3 in., then by Pythagoras' theorem, OM = 4 in.